USE OF STATE PLANE COORDINATES IN ROUTE SURVEYING

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The material in this paper was prepared by Mr. Hugh C. Mitchell, Principal Mathematician (retired) U.S. Coast and Geodetic Survey, as an introduction to the subject of route surveying making use of the State coordinate systems. By using the State plane coordinates in this type of surveying not only can advantage be taken of the horizontal control network of the country with its inherent stability but also can the calculations be carried out by the formulas of plane trigonometry. The subject is discussed in a general way and several examples of typical problems are given.

Comparatively little use has been made of traverse in the solution of route survey problems, but with the increasing availability of coordinates on State-wide grids, and the advantages which the use of such coordinates bring to land and engineering surveys, it seems probable that such use will develop a future importance, much greater than can now be predicted. The field is there and the opportunity also, as the tools, State coordinates, are becoming almost daily more generally available.

The use of plane-rectangular coordinates, be they on a local grid or on a State-wide system, is accomplished by traverse surveying, though there are situations where triangulation may be employed to considerable advantage. These are matters for the surveyor to determine. The present purpose of this study is to show how the coordinates of sought-for points on a route survey may be obtained from coordinate data which describe the fixed conditions controlling the problem.

This purpose will be served by presenting a series of problems, not necessarily typical of those the route survey may bring out, but basic thereto, whose solution will show the way to the solution of specific problems.

Route surveys include surveys for locating or relocating railroads, highways, pipe lines, transmission lines, etc. The employment of State coordinates on such surveys, gives them the qualities of <u>accuracy</u> and <u>permanency</u>, and by making considerable field work unnecessary may be productive of considerable <u>economy</u>. The substitution of office computations for some field surveys is also a great <u>convenience</u>. In route surveys, the use of the random line is largely eliminated by the use of coordinates, though there will be cases where the random line is used as a base line from which side shots determine desired locations on the route, and at the same time, obstacles along the route are avoided.

If a road is to be relocated in part, curves eliminated or reduced, the use of coordinates makes it easy to transfer plans made in the office to positions on the ground. If a new road is to be run direct between points on other roads, and the coordinates of those points are known, a direct line can be computed and if no obstacles intervene, run out on the ground without the use of a random line. If obstacles intervene, but a direct line is to be located, a traverse is run, coordinates of traverse stations determined, from which coordinates of points on the line can be established on the ground. This is a very flexible procedure.

In a road survey, when a curve is reached, the coordinates of the center of a circular curve can be computed from the coordinates of the P.C., the radius of the curve and the azimuth of the tangent. From the center, with the known radius and various azimuths, other points on the curve are determined. These are easily transferred to the ground.

If it is desired to establish 100-foot stations along a curve, the coordinates of the stations are computed from the center, with the given radius and the azimuths to the points obtained with central angles. The central angle will be built up from the azimuth to the P.C.; account is taken of whether arc or chord is used.

A traverse along the route of the curve may provide the simplest method of locating the curve on the ground. With a traverse, obstacles may be dodged and points set between the obstacles. A clear line of sight along the curve will not be necessary. This is an advantage in a preliminary survey. If the total curve is to be composed of a series of simple curves of different radii, the procedure will be about as follows: When the PC is reached, the first curve is established by determining its center, and points along the curve as noted elsewhere. When the PCC is reached, the center of the second curve is computed, using the coordinates of the PCC, the new radius, and the azimuth of the radius of the first curve to this PCC. Points on the second curve are then computed from its center, and so on, through a series of any number of curves, until the PT is reached.

In the following pages, some examples of specific procedure will be described.

A straight line on a grid is defined by (a) the coordinates of its end points, or by (b) the coordinates of one end point and its length and azimuth.

(a) If the coordinates of its end points are known, its length and azimuth are obtained by computation. If the coordinates of the ends of the line AB are, for A, x_1, y_1, and for B, x_2, y_2, its length is

$$L = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and its azimuth is obtained by the formula

$$\text{tangent of azimuth} = \frac{(x_1 - x_2)}{(y_1 - y_2)}.$$

Representing $(x_1 - x_2)$ by Δx and $(y_1 - y_2)$ by Δy, these formulas become

$$\text{length} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\text{tangent of azimuth} = \frac{\Delta x}{\Delta y}.$$

As azimuths are reckoned from south clockwise through 360°, attention must be given the signs of Δx and of Δy.

The following table is useful:

Quadrant	Azimuth	sine	cosine	tangent	Δx	Δy
S.W.	0° - 90°	+	+	+	+	+
N.W.	90° - 180°	+	-	-	+	-
N.E.	180° - 270°	-	-	+	-	-
S.E.	270° - 360°(or 0°)	-	+	-	-	+

(b) If the coordinates of $A(x_1, y_1)$ are known, and the length L of AB and its azimuth, Az, the coordinates of B are obtained by subtracting the values of Δx (= L sin Az) and Δy (= L cos Az) from x_1 and y_1. The subtraction is algebraic, due regard being had for signs.

Considering Δx and Δy as numerical quantities only, the following rules show their application to x_1, y_1 to obtain x_2, y_2:

In S.W. quadrant, subtract Δx and Δy,
In N.W. quadrant, subtract Δx and add Δy,
In N.E. quadrant, add Δx and Δy,
In S.E. quadrant, add Δx and subtract Δy.

Many problems in route surveying concern curves, and the positioning of curves with respect to tangents. In this work, it is often sufficient to know only a point on the tangent and the azimuth of the tangent. The problem most often starts with a tangent or tangents, and determines a curve to fit prescribed conditions. Various problems will be considered. These are intended to show the method of computation, rather than to serve as examples of route layouts.

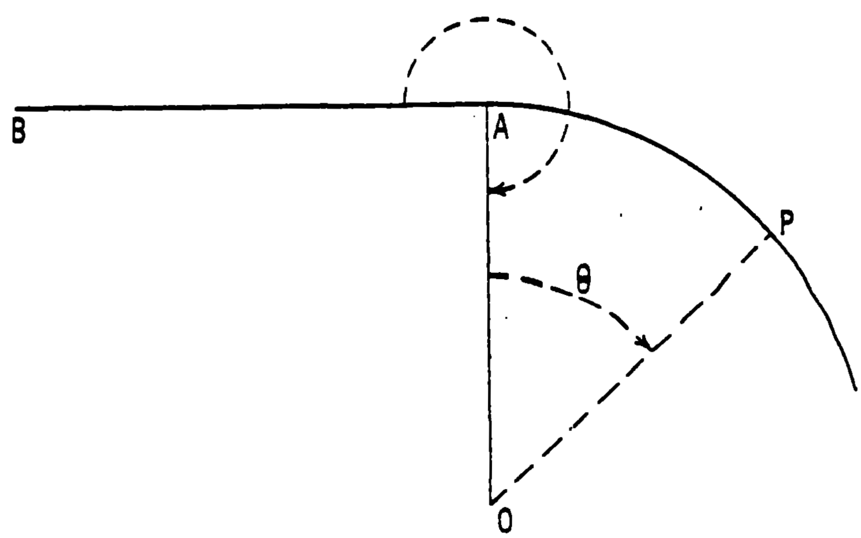
The circle is the simplest curve employed in route surveys. A circle may be defined by the coordinates of its center and the length of its radius. An arc of a circle may be defined by the coordinates of its center and the coordinates of its end points; also by the coordinates of its center, the length of its radius and the azimuths of its limiting radii. Given the coordinates of the center of the circle and the length of its radius, points on its circumference may be computed for each of a series of radial azimuths by applying the Δx and Δy for each of the azimuths to the coordinates of the center.

While each of the following examples is a complete problem, they are arranged in order of increasing complexity, and should be studied in that order. As a general rule, if the problem admits of graphical solution, the computation of coordinates may follow the plan of the graphical solution.

Problem I.

Given the coordinates of a point A, the azimuth of a tangent AB, and the radius R of a circle tangent at A: To define the circle and determine points on it. Two such circles may be constructed, but only one will be illustrated.

<u>Construction</u>: Draw AO = R, perpendicular to AB at A. O is the center of the required circle. With the radius R and center O, the circle may be drawn. Any point, such as P, on the circle may be defined by the angle Θ which the radius to the point makes with the radius AO.



Computation: Compute the azimuth of A-O by adding the angle at A, 270° reckoned in a clockwise direction, between B and O, to the azimuth of A-B. With this azimuth and the length R, compute the Δx and Δy of O relative to A, and apply these to the coordinates of A. This gives the coordinates of O. For any point on the circle, such as P defined by the angle θ, first compute the azimuth of O-A by adding or subtracting 180° to the azimuth A-O. To this azimuth add the angle θ. This gives the azimuth of O-P; coordinates of P are then computed from the coordinates of O.

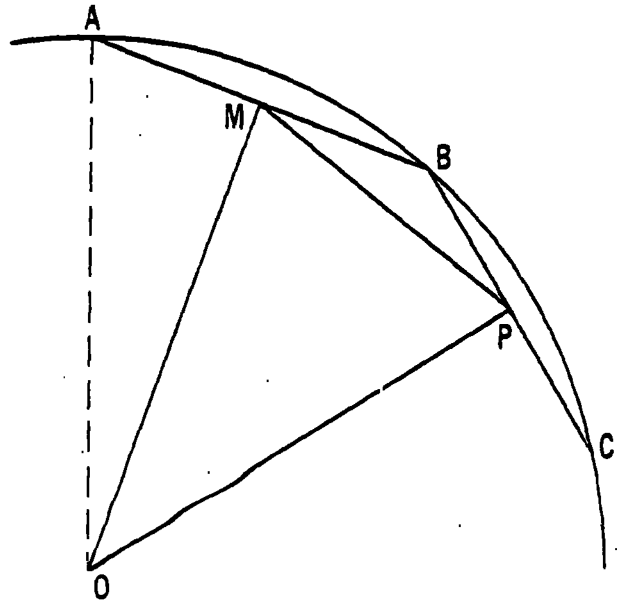
Problem II.

Given the coordinates of the points A, B, and C: To determine and define a circle passing through these three points.

Construction: Draw MO perpendicular to AB at its center point M, and PO perpendicular to BC at its center point P. The point of intersection O, is the center of the required circle. Lines to A, B, and C are equal and are radii of the circle.

Computation: Using coordinates, compute the azimuths of A-B and B-C, and the coordinates of the center points, M and P. Compute the azimuths of M-O and P-O, which are perpendicular to AB and BC. Using coordinates, compute the length and azimuth of MP. In the triangle OMP, the side MP is known, and also the azimuths of the three sides, and therefore the three angles of the triangle.

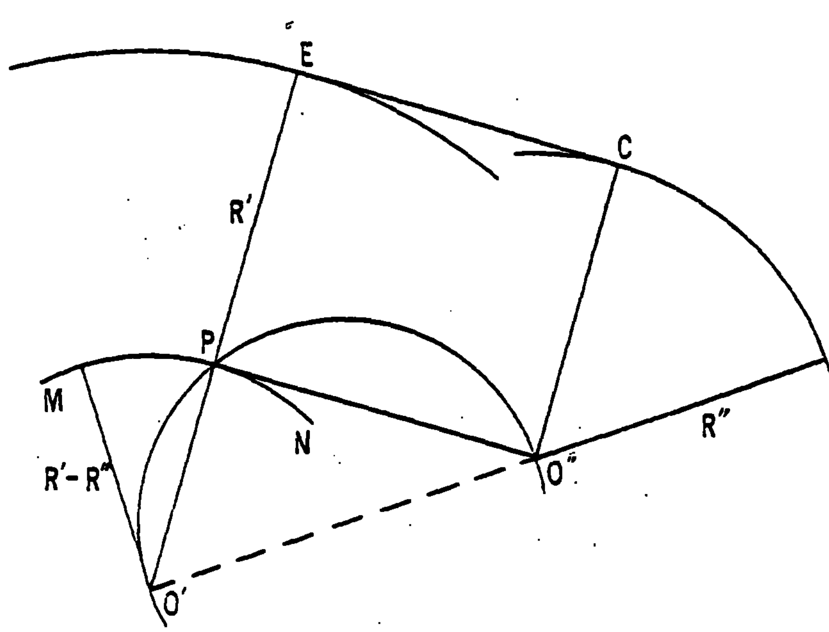
Solve the triangle for the other sides, OM and OP. With these sides and their azimuths, compute the coordinates of O from M and P, obtaining two sets of values which must check. Using the coordinates of O and of A, compute the length of the radius OA. Identical values should be obtained for OB and OC.



The circle can be defined by the coordinates of O and of a point on the circle; by coordinates of O and radius R; and by the coordinates of A, B, and C.

Problem III.

Given the coordinates of the centers and the lengths of radii of two circles: To determine a tangent common to the circles.



<u>Construction</u>: O' and R' define one circle, O" and R" the other. Draw O'D through O" to D on second circle. With O' as center and radius equal to R' - R" draw arc MN, cutting circle constructed on O'O" as a diameter, at point P. Draw PO". The required tangent will be equal to PO" and parallel to it. The point E will be on the radius through P, and C will be on the radius parallel to O'E.

The figure ECO"P is a rectangle. The angles at O" and P are right angles by construction, and the sides EP and CO" are equal to the radius R". The line EC is tangent to the first circle at point E, and to the second circle at point C.

<u>Computation</u>; In the triangle PO"O', the angle at P is a right angle, the side O'P is equal to R' - R", and the length and azimuth of the side O'O" are computed from the coordinates of O' and O". The triangle is solved for the remaining angles and the side PO".

The azimuth of O'-E is obtained with the azimuth of O'-O" and the angle O"O'E. With these data, the coordinates of E are computed from the coordinates of O'. The azimuth of O"-C is the same as for O'-E, and O"C equals R". With these data, the coordinates of C are obtained from the coordinates of O". The coordinates of C can also be obtained from E with the length of EC = PO", and the azimuth of E-C, which is the same as the azimuth of P-O", at right angles to O'E.

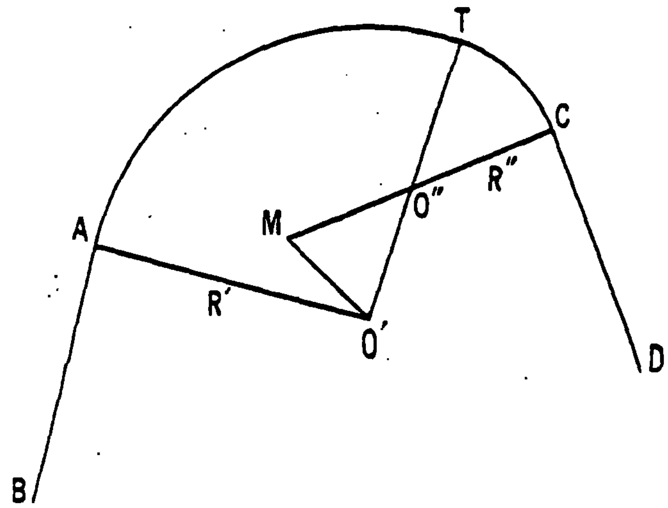
Problem IV.

Given the coordinates of points A and C and azimuths of tangents AB and CD: To determine and define two circular arcs tangent at A and C to the lines AB and CD and tangent to each other.

To obtain a specific solution, the radius of one circle must be known. The value of R', for a circle tangent to AB at A is assumed.

<u>Construction</u>: Plot O', the center of known circle, by laying off AO' equal to R' and at right angles to AB at A. Plot M, by laying off CM equal to R' and perpendicular to CD at C. Connect M and O'. Construct angle MO'T equal to angle CMO' and lay off O'T equal to R'. Then O"M = O"O' and O"T will equal R", the radius of the required circle, O"C.

With R' as radius and center at O' construct arc AT; with O" as center and radius O"C = O"T, construct arc TC. ATC is the required curve.



Computation: From coordinates of A with radius R' and azimuth 90° from azimuth of A-B, compute coordinates of O'.

From coordinates of C, with R' and azimuth 90° from azimuth of C-D, compute coordinates of M.

From coordinates of O' and of M, compute length and azimuth of O'M.

Obtain angle CMO' from azimuths. Make angle MO'T equal to angle CMO'.

Solve triangle O"O'M for side O"M = O"O'.

Compute coordinates of O" from O' and, for check, from M.

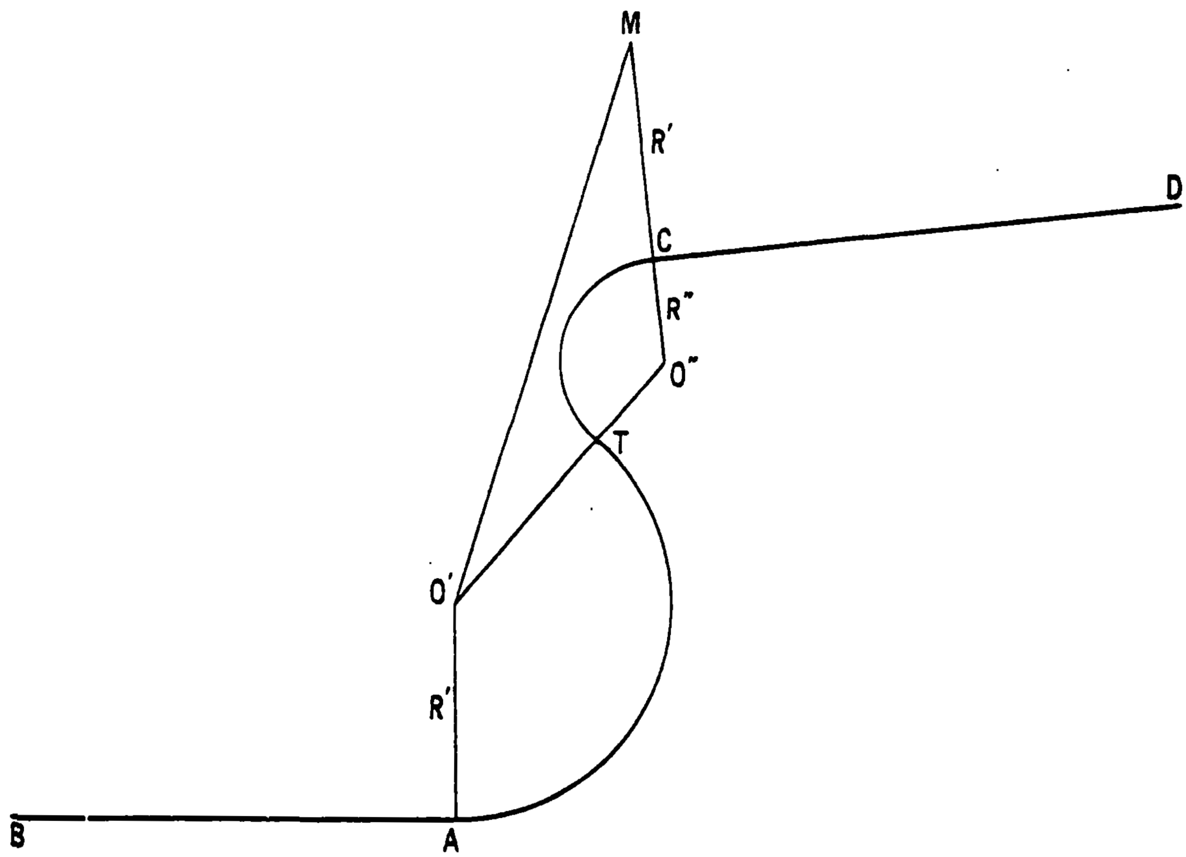
Compute coordinates of T from O', with radius R' and azimuth of O'O". Check coordinates of T by computation from O" with R" and azimuth O'O".

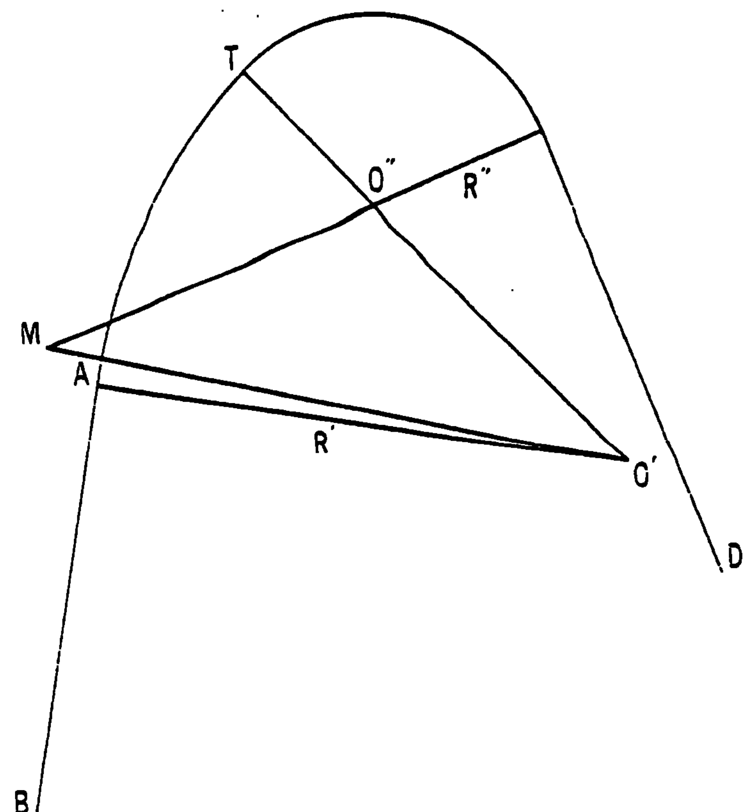
O' is center of arc AT.

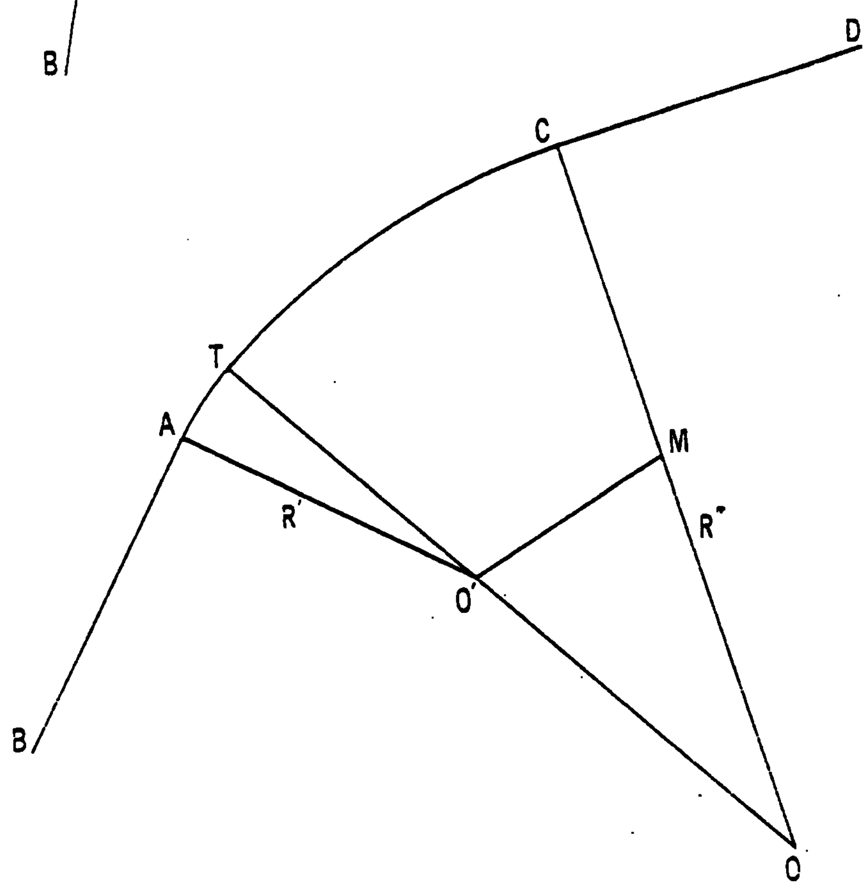
O" is center of arc TC.

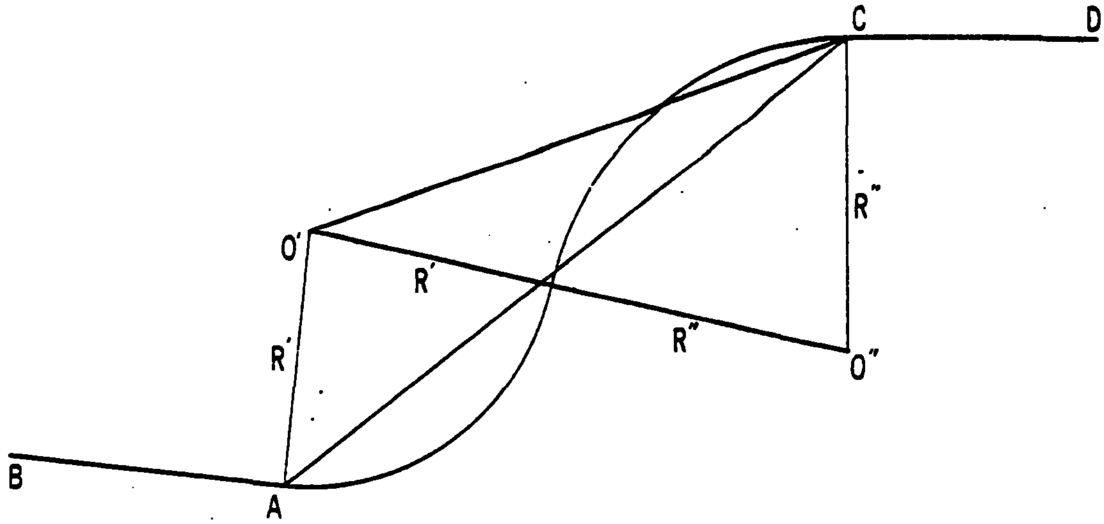
Coordinates of points on these arcs can be computed from their centers with appropriate radii (R' or R") and selected azimuths.

There are many variations of this problem, depending upon the relative positions of the points A and C, the azimuths of the tangents, and the value adopted for R'. A few examples are shown in the following figures. These are lettered to conform to the one demonstrated.




T
O''
R''
M
A
R'
O'
D
B


D
C
T
A
M
R'
R''
O'
B
O



Given the coordinates of A and C, and azimuths of AB and CD: To determine two circles connecting these tangents.

To obtain a specific solution, one radius must be known. Assume R' as known. This fixes the solution, as follows: In the triangle CO"O', we have the side O'C, whose length and azimuth are computed from coordinates of O' and C. We also have the $\angle$ O'CO", the difference of the azimuths of CO' and CO" (perpendicular to CD). The side O"O' is the sum (R" + R'), R' known, R" sought.

Then

$$(R'' + R')^2 = R''^2 + (CO')^2 - 2\,R''\cdot CO'\cdot \cos\angle O'CO''$$

$$R''^2 + 2\,R''R' + R'^2 = R''^2 + (CO')^2 - 2R''(CO')\cos\angle O'CO''$$

$$R''\left[2R' + 2(CO')\cos\angle O'CO''\right] = (CO')^2 - R'^2$$

$$R'' = \frac{(CO')^2 - R'^2}{2\,R' + 2(CO')\cos\angle O'CO''}$$

Although this discussion has been quite limited in scope and has barely scratched the surface in the way of problems to be solved, it is hoped that the engineer may see the advantages which may obtain from making use of the State plane coordinate systems in his route surveying.

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