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Algorithms For Confidence Circles and Ellipses

Wayne E. Hoover

Charting and Geodetic Services
Rockville, MD
September 1984

U. S. DEPARTMENT OF COMMERCE
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ERRATA

Hoover, Wayne E., "Algorithms for Confidence Circles and Ellipses." Washington, D.C., National Oceanic and Atmospheric Administration. NOAA Technical Report NOS 107 C&GS 3, (1984) pp 1-29.

1. Page 8, In the equation for p(R), insert a closing bracket between the superscript 2 and the dxdy.

2. Page 13, Equation 7b incorrectly reads:

   $$p = [\tfrac{1}{2} + T_1 + T_2 + T_3]c/n$$

   The correct formula (for the trapezoidal rule) is:

   $$p = [\tfrac{1}{2}T_1 + T_2 + T_3]c/n$$

3. Page 25, Line 19 erroneously contains "erors" the correct spelling is "errors".

3-27-86
WAYNE E. HOOVER

ALGORITHMS FOR CONFIDENCE CIRCLES AND ELLIPSES

Abstract

In many hydrographic surveying, navigation, and position location systems the observed position is defined as the intersection of two lines of position, each of which may be in error. This paper gives algorithms with new stopping criteria for the determination of the probability that the true position T lies within a circle of given radius centered at the observed position O, and conversely, the determination of the radius of a circle C with center O such that the probability is p that T lies within C. In either case, the circle centered at O is called a confidence circle.

Confidence ellipses are also considered and are shown to be superior to confidence circles since they provide the same probability of location but generally over a significantly smaller region.

It is assumed that the errors associated with the lines of position may be approximated by a nonorthogonal bivariate dependent Gaussian distribution where the errors are measured orthogonally to the lines of position. The algorithms given are straightforward and easy to implement on a microcomputer.

Biographical Sketch of Wayne E. Hoover

Wayne E. Hoover is a systems analyst with the National Oceanic and Atmospheric Administration in Woods Hole, Massachusetts, and also is an adjunct professor of mathematics at Cape Cod Community College in West Barnstable. In 1977 he received his Ph.D. in numerical analysis from Michigan State University.

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ILLUSTRATIONS

TABLES

KEY TO SYMBOLS

		Page
C	Confidence circle	8
c	Ratio of smaller to larger standard deviation: σ_y/σ_x	8
CEP	Circular error probable = radius of the 50% confidence circle	16
E	Probability error estimate	11, 13
e	2.7182818284590452...	7
$f(\phi)$	Integrand	9
g_i	Auxiliary function	14
h	Integration step size	10, 13
K	Ratio: R/σ_x	8
K_i	ith iterate of K	14
k	Elliptical scale factor	7
$k\sigma_x$	Length of semimajor axis of confidence ellipse	7
$k\sigma_y$	Length of semiminor axis of confidence ellipse	7
L_1	First line of position	2
L_2	Second line of position	2
LOP	Line of position	1
n	Positive integer	10, 13
O	Observed position	1
p	Probability associated with confidence circle or ellipse	7, 8, 13
p(R)	Probability as a function of radius of confidence circle	8, 13
p(K,c)	Circular error probability	9
R	Radius of confidence circle	8, 14

		Page
$R(p)$	Radius of confidence circle as a function of probability	11, 14
T	True position	1
T_1	End point trapezoidal sum	13
T_2	Interior end point trapezoidal sum	13
T_3	Centroid or midpoint sum	13
u_1	First nonorthogonal axis	2
u_2	Second nonorthogonal axis	2
$w(\phi)$	Auxiliary function	9
x	First orthogonal axis	3
y	Second orthogonal axis	3
α	Angle of crossing, $0 < \alpha < \pi$	2
β	$2c/\pi$	9
γ	$(K/2c)^2$	9
θ	Orientation of semimajor axis of error ellipse with respect to L_1	3, 12
π	3.14159265358979 32...	2
ρ_{12}	Correlation coefficient in u_1-u_2 coordinate system	3
ρ_{xy}	Transformed correlation coefficient	6
σ_1	Standard deviation associated with L_1	3
σ_2	Standard deviation associated with L_2	3
σ_x	Length of semimajor axis of error ellipse	5
σ_y	Length of semiminor axis of error ellipse	5
ϕ	Variable of integration	9
1dRMS	Radial or root mean square error	15
2dRMS	Upper bound for radius of 95% confidence circle	15

ALGORITHMS FOR CONFIDENCE CIRCLES AND ELLIPSES

1.0 INTRODUCTION

Hydrographic surveyors, navigators, and others concerned with position location have traditionally determined their position by means of two intersecting lines of position (LOPs). The LOPs may be derived from celestial observations, trilateration, LORAN signals, satellite signals, etc.

Two questions important to position locators are the following: (1) What is the probability that the true position T, which is generally unknown, is located R units or less from the observed position O; and conversely, (2) What is the radius of the circle C centered at O such that the probability is p that T lies within C.

In either case, a circle of radius R which is centered at the observed position O is called a confidence circle. It is also called a circle of uncertainty or circle of equivalent probability.

This paper will outline the mathematical aspects of these problems and then give new algorithms for their solution. The algorithms are straightforward, efficient, and readily implemented on a microcomputer.

Also, mention will be made of confidence ellipses which are actually much easier to calculate than confidence circles; moreover, they are superior to confidence circles since they provide the same probability of location over a generally significantly smaller area.

Finally, several numerical examples illustrating the application of the algorithms will be presented.

2.0 MATHEMATICAL CONSIDERATIONS

2.1 Geometry

Designate the two lines of position by L_1 and L_2, respectively, and let α, $0 < \alpha < \Pi$, be the crossing angle from L_1 measured in a positive or counterclockwise direction to L_2. Let O denote the intersection of the LOPs. Thus O represents the observed or measured position.

Define the nonorthogonal u_1-u_2 coordinate system such that u_1 and u_2 intersect at O, u_1 is perpendicular to L_1, u_2 is perpendicular to L_2, and the positive angle from u_1 to u_2 is $\Pi + \alpha$. This geometry follows that of Swanson [9] and is illustrated in Figure 1.

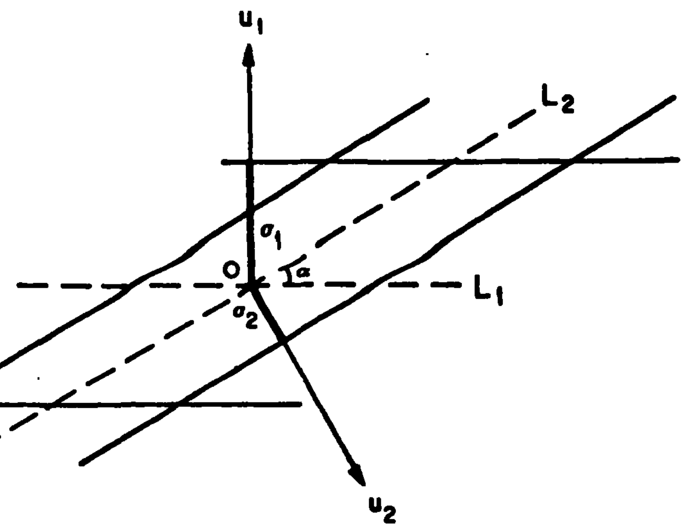

Figure 1 Nonorthogonal Coordinate System

Note that Burt, Kaplan, and Keenly, et al [4] and also Bowditch [2] use a different geometry by reversing the direction of the u_2-axis. In this case, the positive angle from u_1 to u_2 equals α. Moreover, this changes the sign of the correlation coefficient ρ_{12}.

2.2 Assumptions

Throughout this paper we will assume the following:

1. In a small region G containing O, the earth is flat and the two LOPs are straight lines.
2. The errors in the measurements which determine L_1 and L_2 are normally distributed random variables with correlation coefficient ρ_{12}, zero means, and standard deviations σ_1 and σ_2, respectively, where σ_i is measured along u_i which is perpendicular to L_i.
3. The bivariate error distribution is constant throughout the region G.

Thus it is assumed that the errors in the measurements of the LOPs, which may or may not be correlated, may be approximated by a nonorthogonal bivariate dependent Gaussian distribution.

This paper applies only to those position location systems for which the above three assumptions provide the basis for a valid error model. It can be a sizeable task to decide whether this model is appropriate for a specific position location system.

2.3 Transformation to an Orthogonal System

Now transform the nonorthogonal u_1-u_2 system to an orthogonal x-y Cartesian coordinate system centered at O and oriented such that the angle from L_1 to the positive x-axis is given by θ. Following convention, a positive angle is measured in a counterclockwise direction. These coordinate systems are illustrated in Figure 2.

The transformation is given by

$$u_1 = x\sin(\theta) + y\cos(\theta)$$

$$u_2 = x\sin(\alpha - \theta) - y\cos(\alpha - \theta).$$

The angle θ, which is defined in the next section, is chosen so that the transformed variables are stochastically independent.

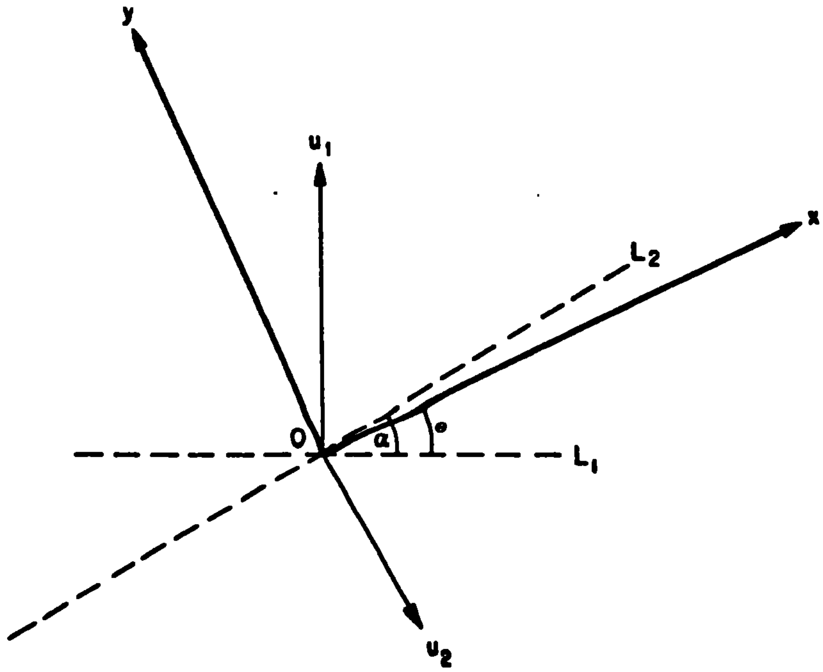

Figure 2 Orthogonal Coordinate System

2.4 The Error Ellipse

In order to determine the radius R(p) of the confidence circle C, or the probability p(R) associated with C, it is necessary to first calculate the parameters of the error ellipse, namely, the lengths of the semimajor and semiminor axes and their orientation with respect to a coordinate system.

Defining the ancillary variables

$$a_1 = \sigma_1^2\sin(2\alpha) + 2\rho_{12}\sigma_1\sigma_2\sin(\alpha)$$

$$a_2 = \sigma_1^2\cos(2\alpha) + 2\rho_{12}\sigma_1\sigma_2\cos(\alpha) + \sigma_2^2$$

$$a_3 = \sigma_1^2 + 2\rho_{12}\sigma_1\sigma_2\cos(\alpha) + \sigma_2^2$$

$$a_4 = [a_1^2 + a_2^2]^{\frac{1}{2}}$$

$$a_5 = 2\sin^2(\alpha)$$

and using the transformation given in Section 2.3, it can be shown that the given nonorthogonal standard deviations σ_1 and σ_2 are transformed to σ_x and σ_y, respectively, in the orthogonal x-y Cartesian coordinate system, where

$$\sigma_x = [(a_3 + a_4)/a_5]^{\frac{1}{2}}$$

$$\sigma_y = [(a_3 - a_4)/a_5]^{\frac{1}{2}}$$

and $\sigma_x \geq \sigma_y$ holds for all valid values of the input variables: $\sigma_1 \geq 0$, $\sigma_2 \geq 0$, $0 < \alpha < \Pi$, and $-1 < \rho_{12} < 1$. The error ellipse is the ellipse with center 0, semimajor axis σ_x which coincides with the positive x-axis, and semiminor axis σ_y which coincides with the positive y-axis.

The orientation of the error ellipse is calculated from

$$\tan(2\theta) = a_1/a_2.$$

Note that this calculation must be performed so that θ is obtained in the proper quadrant. This can be achieved with the aid of the double argument arctangent function or the rectangular-to-polar function. Thus, $-\Pi/2 < \theta < \Pi/2$, where θ is the angle from L_1 to the positive x-axis. As before, a positive angle represents a counterclockwise direction.

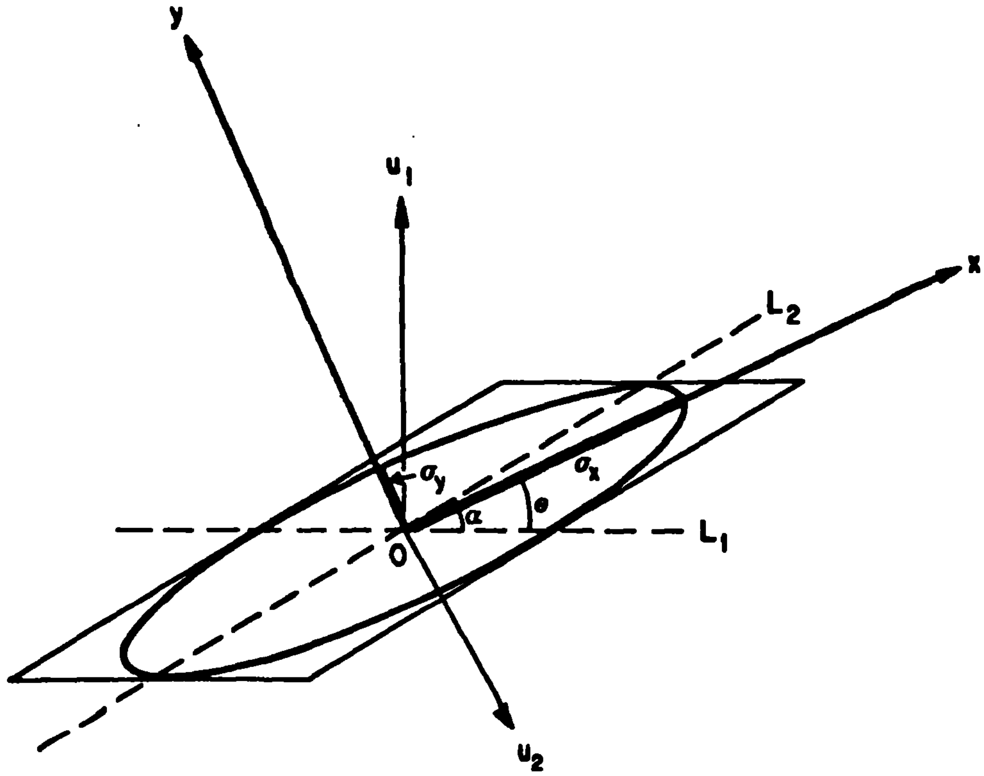

Figure 3 The Error Ellipse

The error ellipse with parameters σ_x, σ_y, and θ is illustrated in Figure 3. In the x-y coordinate system, the correlation ρ_{xy} between the transformed variables is zero.

2.5 Confidence Ellipses

A confidence ellipse is an ellipse which is concentric to the error ellipse and which has parameters $k\sigma_x$, $k\sigma_y$, and θ; k is called the elliptical scale factor.

Since σ_x and σ_y represent the standard deviations of stochastically independent random variables, the addition theorem for the chi-square distribution may be used to show that the probability associated with a confidence ellipse is given by

$$p = 1 - e^{-\frac{1}{2}k^2}.$$

Conversely, the semimajor $k\sigma_x$ and semiminor $k\sigma_y$ axes of a confidence ellipse having specified probability p may be calculated from σ_x, σ_y, and

$$k = [-2*\ln(1 - p)]^{\frac{1}{2}}.$$

Thus the error ellipse is a confidence ellipse with elliptical scale factor k = 1 and probability approximately p = 0.3935. The 50% and 95% confidence ellipses have elliptical scale factors approximately 1.1774 and 2.4477, respectively.

Figure 4 contains a graph of the elliptical scale factor as a function of probability.

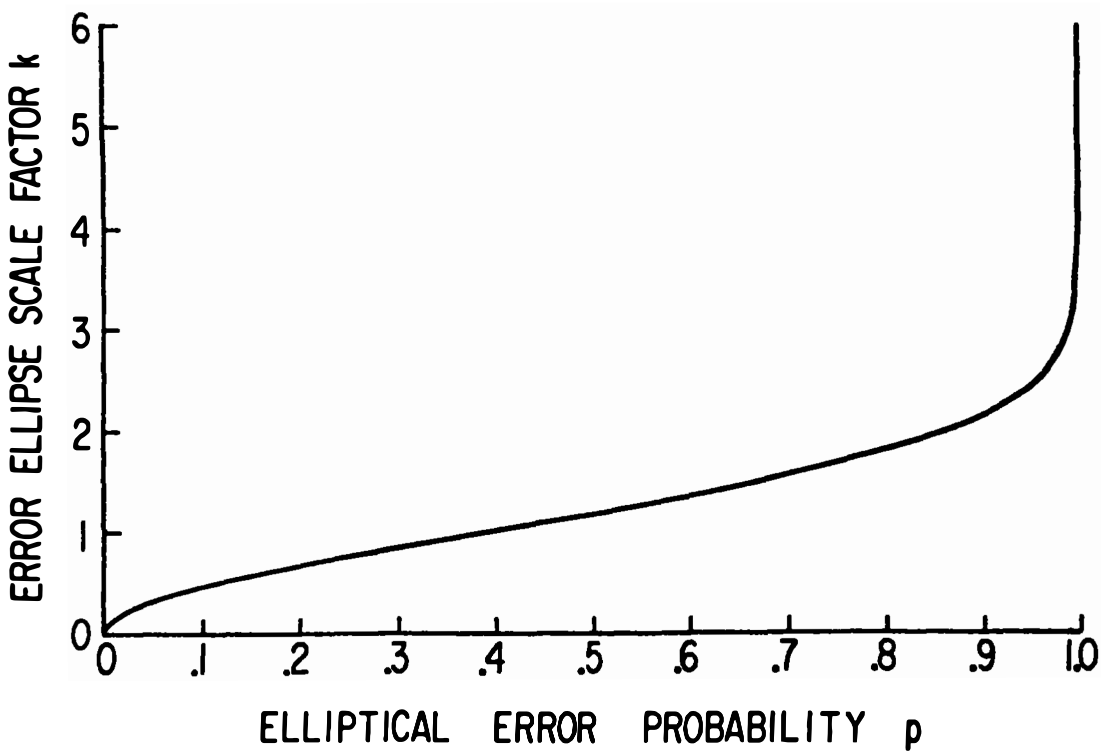

Figure 4 Elliptical Scale Factor vs. Probability

2.6 Confidence Circles

Let C denote a confidence circle, $x^2 + y^2 = R^2$, which is centered at 0 and which has positive radius R. Then the probability $p = p(R)$ that the true position T lies within a confidence circle C is

$$p(R) = \frac{1}{2\pi\sigma_x\sigma_y} \iint_C e^{-\frac{1}{2}\left[\left(\frac{x}{\sigma_x}\right)^2 + \left(\frac{y}{\sigma_y}\right)^2\right]} dxdy.$$

Defining the auxiliary parameters

$K = R/\sigma_x$

$c = \sigma_y/\sigma_x$

$\beta = 2c/\pi$

$\gamma = (K/2c)^2$

and the functions

$$w(\phi) = (c^2 - 1)\cos(\phi) - (c^2 + 1)$$

$$f(\phi) = [e^{\gamma w(\phi)} - 1]/w(\phi)$$

it can be shown that this double integral over the circle C can be reduced to the single definite integral

$$p(R) = p(K,c) = \beta \int_0^{\pi} f(\phi)d\phi .$$

The value of this integral provides the solution to question (1) stated in the introduction.

2.7 Numerical Quadrature

Values of p(K,c) have been tabulated and are given in the Appendix. However, in order to use such a table, double interpolation is required. For values more precise than those given in the table, the integral p(K,c) must be evaluated numerically since it apparently cannot be expressed in closed form. For definite integrals of the type p(K,c), Fettis [6] has shown that if a sufficiently small step size is chosen, the trapezoidal rule provides an estimate for p(R) with arbitrarily small error.

Whenever the trapezoidal rule is effectively employed, Frame [7] suggests that linear combinations of the rule with different step sizes will provide additional estimates to the definite integral with only a minimal increase in computational effort. Such a numerical quadrature formula is the fifth-order derivative corrected Simpson's rule [10] with step size $h = (b - a)/n$:

$$\int_a^b f(x)dx = \frac{h}{30}\left[7[f(a) + f(b)] + 14\sum_{i=1}^{n-1} f(a+ih) + 16\sum_{i=1}^{n} f(a+ih-h/2)\right] - \frac{h^2}{60}[f'(b) - f'(a)].$$

Since the integrand, $f(\phi) = [e^{\gamma w(\phi)} - 1]/w(\phi)$, which is required for the calculation of p(R), is periodic with period 2π, is symmetric about π, and has continuous first derivative, $f'(\phi)$ vanishes at the end points of the interval of integration, $\phi = 0$ and $\phi = \pi$. Therefore, for the definite integral under consideration, the derivative corrected Simpson's rule is a linear combination of trapezoidal sums with step sizes $h = \pi/n$ and $h = \pi/(2n)$.

The solution to question (1) stated in the introduction may now be obtained by applying the trapezoidal rule with step size $\pi/(2n)$ to p(K,c), constructing from appropriate trapezoidal sums the derivative corrected Simpson's value, and then using the absolute value of the difference to estimate the maximum absolute error for the former calculation which is taken as an approximation to the probability p(R).

The technique of employing the fifth-order derivative corrected Simpson's rule in order to estimate the error, E, in the numerical quadrature is believed to be new and is more efficient than the customary repeated halving the step size until a sufficiently small difference is obtained, since the traditional technique uses a quadrature formula of the same order to approximate the error. This technique is well suited to microcomputers where time is more critical than on larger computer systems.

The required inputs for the calculation of the probability p(R) are σ_1, σ_2, α, ρ_{12}, R, and n. The value of n is chosen so that the error estimate E is sufficiently small. In most practical applications (i.e., $K \leq 4$ and $c \geq 0.1$), a value of n = 20 will result in at least seven digit accuracy for p(R).

Question (2) stated in the introduction may now be solved by iterating on the radius R(p) until the desired probability is obtained. In practice, the iteration is actually on the auxiliary parameter $K = R/\sigma_x$. For values of p(R) less than 0.9999999, K assumes values between zero and 5.7.

These considerations provide an outline of the theoretical foundation for the two algorithms given in the next section for the calculation of p(R) and R(p) associated with confidence circles.

The calculations required for the parameters of a confidence ellipse are straightforward and have been given in Sections 2.4 and 2.5.

3.0 ALGORITHMS

The first algorithm solves question (1) and is also referenced by the second algorithm. The calculation of θ in step two is an optional calculation since it is not required for the computation of the probability p(R) associated with a confidence circle. The input parameters are σ_1, σ_2, α, ρ_{12}, and R.

The second algorithm solves question (2) and is based on the secant method. Note that the iteration is actually performed on K which is related to the radius of a confidence circle by $K = R/\sigma_x$. The input parameters are σ_1, σ_2, α, ρ_{12}, and p.

3.1 Algorithm 1 for p(R)

1. $a_1 = \sigma_1^2 \sin(2\alpha) + 2\rho_{12}\sigma_1\sigma_2 \sin(\alpha)$

   $a_2 = \sigma_1^2 \cos(2\alpha) + 2\rho_{12}\sigma_1\sigma_2 \cos(\alpha) + \sigma_2^2$

   $a_3 = \sigma_1^2 + 2\rho_{12}\sigma_1\sigma_2 \cos(\alpha) + \sigma_2^2$

   $a_4 = [a_1^2 + a_2^2]^{\frac{1}{2}}$

   $a_5 = 2\sin^2(\alpha)$

2. $\sigma_x = [(a_3 + a_4)/a_5]^{\frac{1}{2}}$

   $\sigma_y = [(a_3 - a_4)/a_5]^{\frac{1}{2}}$

   $\theta = \frac{1}{2}\arctan(a_1/a_2)$ (Note: use arctan (y,x) or P-R function)

$c = \sigma_y/\sigma_x$

3. $K = R/\sigma_x$

4. $\gamma = (K/2c)^2$

5. Select a positive integer n (e.g., n = 4)

6. $h = \pi/n$

$w(\phi) = (c^2 - 1)\cos(\phi) - (c^2 + 1)$

$f(\phi) = [e^{\gamma w(\phi)} - 1]/w(\phi)$

$T_1 = f(0) + f(\pi)$

$T_2 = \sum_{i=1}^{n-1} f(ih)$

$T_3 = \sum_{i=1}^{n} f[(i - \frac{1}{2})h]$

7. $E = [[T_1 + 2(T_2 - T_3)]c/n]^2/6$

$p = [\frac{1}{2} + T_1 + T_2 + T_3]c/n$

8. If E is sufficiently small (e.g., $E < 10^{-5}$), accept $p = p(R)$. Otherwise, select a larger value for n and repeat steps 6 through 8.

3.2 Algorithm 2 for R(p)

1. Perform steps 1 and 2 of algorithm 1.
2. Set $i = 1$ and select appropriate starting values for the secant method (e.g. $K_0 = 0.1$, $K_1 = 3.9$, $g_0 = 0.08 - p$, and $g_1 = 1.0$).
3. Using the value of

$$K_{i+1} = K_i - g_i \frac{K_i - K_{i-1}}{g_i - g_{i-1}}$$

   where $g_i = p_i - p$ for $i > 1$, perform steps 4 through 8 of algorithm 1 to obtain probability p_{i+1}.
4. If g_{i+1} is sufficiently small (e.g., $|g_{i+1}| < 10^{-7}$), set $R = R(p) = \sigma_x K_{i+1}$ and stop. Otherwise, repeat steps 3 and 4 with i replaced by $i + 1$.

4.0 NUMERICAL EXAMPLES

The following examples illustrate the application of the two algorithms presented in the previous section.

4.1 Example 1

A navigator reports the ship's position at 41°46' N and 50°14' W. Assuming the angle of crossing between the two LOPs is $\alpha = 30°$, there are no systematic errors, and the random errors in the two nonorthogonal directions are normally and independently distributed with standard deviations $\sigma_1 = 2$ nm and $\sigma_2 = 1$ nm, calculate the parameters of the error ellipse and the radii of the confidence circles for the various probabilities indicated in Table 1. Also compute the sizes, areas, and the probabilities associated with the 1dRMS and 2dRMS circles where $1dRMS^2 = \sigma_x^2 + \sigma_y^2$ and 2dRMS = 2*1dRMS. The term 1dRMS is also called radial error or root mean square error.

Use algorithm 1 to calculate the parameters of the error ellipse: σ_x = 4.3778 nm, σ_y = 0.9137 nm, and θ = 24.5533°. The resulting error ellipse is shown in Figure 5.

Continuing with algorithm 1, set n = 7 and compute 2dRMS = 8.9443 nm, p(2dRMS) = 0.9579, and $E = 4.3*10^{-7}$ where E is an estimate of the maximum absolute error in p(R). Similarly calculate the values for the 1dRMS circle as indicated in Table 1.

σ_1	2
σ_2	1
α	30
σ_x	4.38
σ_y	.91
θ	24.55
σ_y/σ_x	.21
1dRMS	4.47
2dRMS	8.94

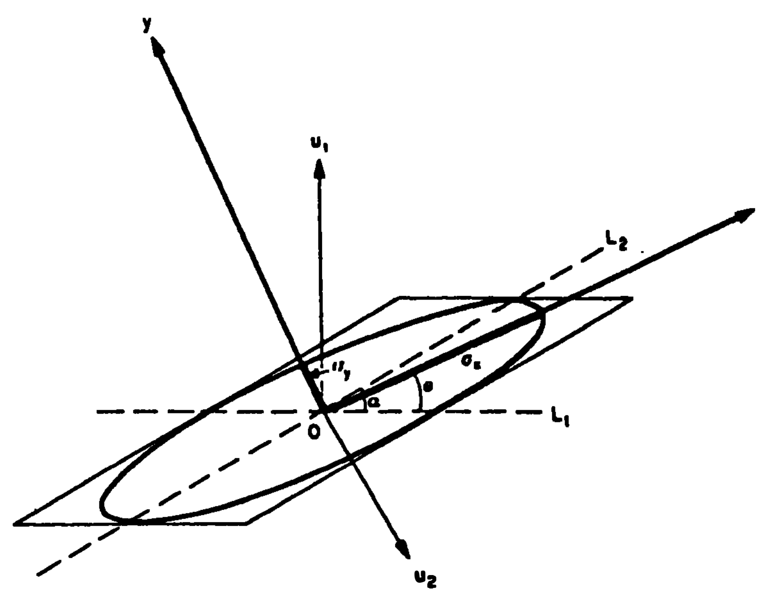

Figure 5 The Error Ellipse

Next, use algorithm 2 to calculate the remaining values listed in Table 1. For reference, the 1dRMS and 2dRMS values computed from algorithm 1 are included in Table 1. See Figure 6 for a plot of the radius of the confidence circle as a nonlinear function of probability. Note that 2dRMS is an upper bound for the radius of the 95% circle. The circular probable error or circular error probable, CEP, is the radius of the 50% circle.

Use the elliptical scale factor k = 2.4477 to calculate the semimajor and semiminor axes of the 95% ellipse, 10.7158 nm and 2.2365 nm, respectively. The area of the 95% ellipse is 75.3 nm^2. Since the radius of the 95% circle is 8.6302 nm, the area of the 95% circle is 234.0 nm^2. Thus the area of the 95% circle is 211% larger than the area of the 95% ellipse and yet both provide the same confidence for position location.

Table 1 Parameters associated with $\sigma_1 = 2$, $\sigma_2 = 1$, $\alpha = 30°$, and $\rho_{12} = 0$

Probability p	Radius R	Area A	n	Error Bound E
.01	0.2846	0.3	1	1.3E-13
.10	0.9565	2.9	1	1.9E-7
.50	3.1033	30.3	3	6.2E-8
.68218	4.4721*	62.8	4	1.8E-7
.75	5.1216	82.4	5	1.8E-8
.90	7.2604	165.6	6	3.4E-7
.95	8.6302	234.0	7	2.5E-7
.95786	8.9443**	251.3	7	4.3E-7
.99	11.3144	402.2	8	6.1E-7
.999	14.4349	654.6	9	4.0E-7
.9999	17.0573	914.1	10	1.0E-7
.99999	19.3592	1177.4	10	1.1E-7

* 1dRMS
** 2dRMS

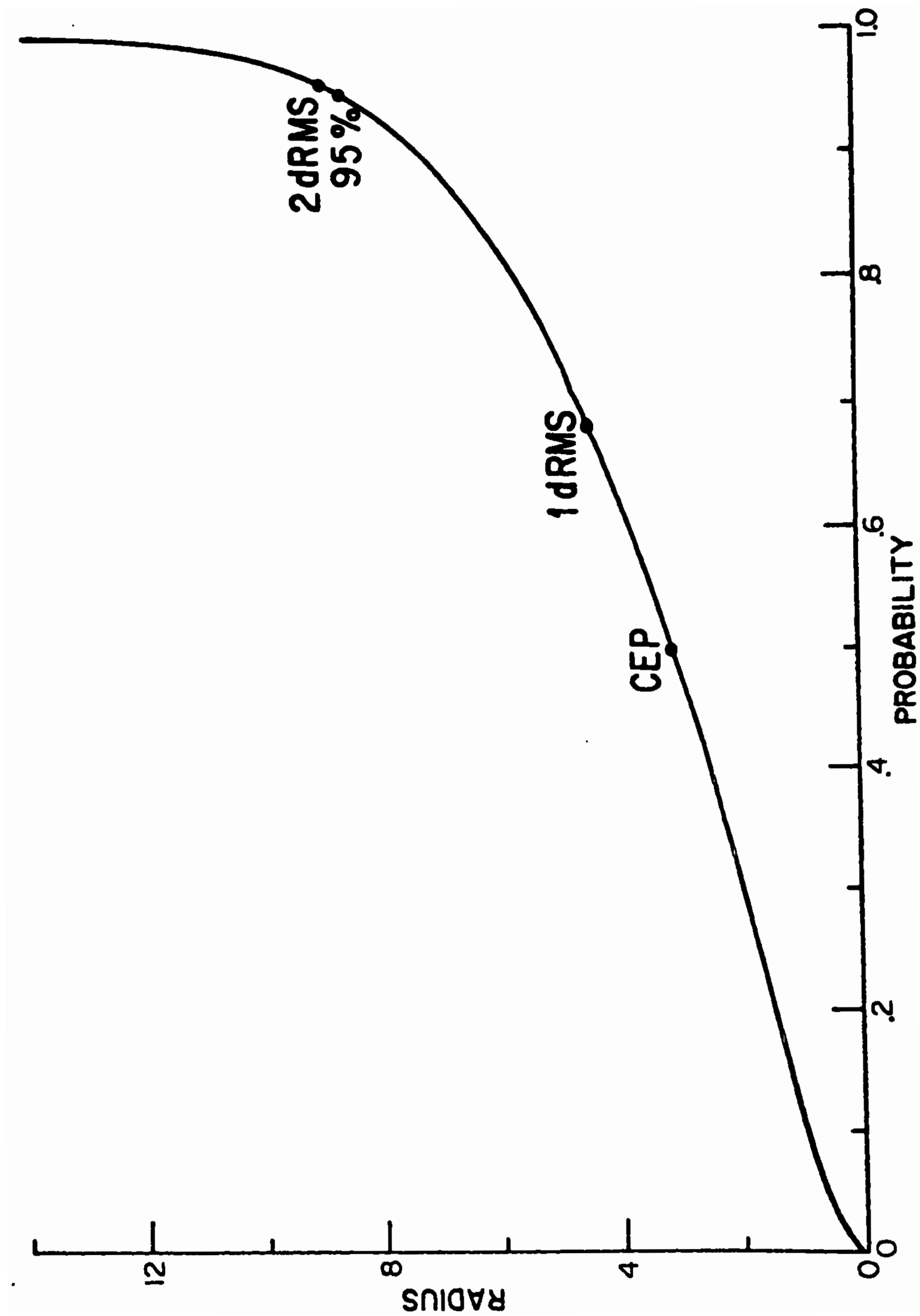

Figure 6 Confidence Circles with $\sigma_1 = 2$, $\sigma_2 = 1$, $\alpha = 30°$, and $\rho_{12} = 0$

4.2 Example 2

Consider a position location system where $\sigma_1 = \sigma_2 = 1$ unit and the angle of crossing α varies between 0 and Π. For various values of α and assuming $\rho_{12} = 0$, Table 2 gives the parameters of the error ellipse and the sizes, areas, and probabilities associated with the 95% and 2dRMS confidence circles. Table 3 gives the areas of these 95% circles and ellipses as a function of α. Figure 7 shows a plot of the radius of the 95% circle as a function of α.

4.3 Example 3

(See Bowditch [2].) Assuming $\sigma_1 = 15$ m, $\sigma_2 = 20$ m, $\alpha = 50°$, and $\rho_{12} = 0$, determine the probability of location within a circle of radius R = 30 m.

Set n = 2 in algorithm 1 and obtain $\sigma_x = 29.8895$ m, $\sigma_y = 13.1023$ m, $\theta = 15.7733°$, and p(30 m) = 0.6175. The error estimate is $E = 1.3*10^{-7}$ while the actual error is $1.0*10^{-8}$.

Set n = 3 in algorithm 2 and compute the radius of the 95% circle: R = 60.2437 m with $E = 1.4*10^{-6}$. Also, using n = 5, the radius of the 99.9% circle is found to be R = 99.3274 m with $E = 8.1*10^{-9}$.

The parameters of the 95% ellipse are $k\sigma_x = 73.1620$ m, $k\sigma_y = 32.0712$ m, and $\theta = 15.7733°$. The area of the 95% circle, 11401.8 m^2 is 55% greater than the area of the 95% confidence ellipse, 7371.4 m^2.

Table 2 Parameters associated with $\sigma_1 = \sigma_2 = 1$ and $\rho_{12} = 0$

α	θ	σ_x	σ_y	$c=\sigma_y/\sigma_x$	Radius 2dRMS	Area R=2dRMS	Prob p(R=2dRMS)	Radius R(p=0.95)	Area R(p=0.95)	$\frac{2dRMS}{R(p=0.95)}$
0.1 (179.9)	0.05 (-.05)	810.2848	.70711	.00087	1620.5702	8250 600.6	.95450	1588.1292	7923 581.2	1.0199
1 (179)	0.5 (-.50)	81.0295	.70713	.0087	162.0652	82 514.3	.95451	158.8165	79 239.4	1.0205
5 (175)	2.5 (-2.5)	16.2108	.70778	.0437	32.4526	3 308.6	.95465	31.7805	3 173.0	1.0211
10 (170)	5 (-5)	8.1131	.7098	.0875	16.2883	833.5	.95511	15.9174	796.0	1.0233
20 (160)	10 (-10)	4.0721	.7180	.1763	8.2698	214.9	.95693	8.0140	201.8	1.0319
30 (150)	15 (-15)	2.7321	.7321	.2679	5.6569	100.5	.95986	5.4069	91.8	1.0462
40 (140)	20 (-20)	2.0674	.7525	.3640	4.4003	60.8	.96375	4.1280	53.5	1.0660
50 (130)	25 (-25)	1.6732	.7802	.4663	3.6922	42.8	.96833	3.3867	36.0	1.0902
60 (120)	30 (-30)	1.4142	.8165	.5774	3.2660	33.5	.97316	2.9266	26.9	1.1160
70 (110)	35 (-35)	1.2328	.8632	.7002	3.0099	28.5	.97753	2.6458	22.0	1.1376
80 (100)	40 (-40)	1.1001	.9231	.8391	2.8721	25.9	.98059	2.4950	19.6	1.1511
90	45	1.0000	1.0000	1.0000	2.8284	25.1	.98168	2.4477	18.8	1.1555

Table 3 Areas of 95% Confidence Circles and Ellipses when $\sigma_1 = \sigma_2 = 1$ and $\rho_{12} = 0$

α	σ_x	σ_y	A_1=Area of 95% Circle	A_2=Area of 95% Ellipse	$\frac{A_1}{A_2}$
0.1 (179.9)	810.2848	.70711	7923 581.2	10 784.6	734.71
1 (179)	81.0295	.70713	79 239.4	1 078.5	73.47
5 (175)	16.2108	.70778	3 173.0	216.0	14.69
10 (170)	8.1131	.7098	796.0	108.4	7.34
20 (160)	4.0721	.7180	201.8	55.0	3.67
30 (150)	2.7321	.7321	91.8	37.6	2.44
40 (140)	2.0674	.7525	53.5	29.3	1.83
50 (130)	1.6732	.7802	36.0	24.6	1.47
60 (120)	1.4142	.8165	26.9	21.7	1.24
70 (110)	1.2328	.8632	22.0	20.0	1.10
80 (100)	1.1001	.9231	19.6	19.1	1.02
90	1.0000	1.0000	18.8	18.8	1.00

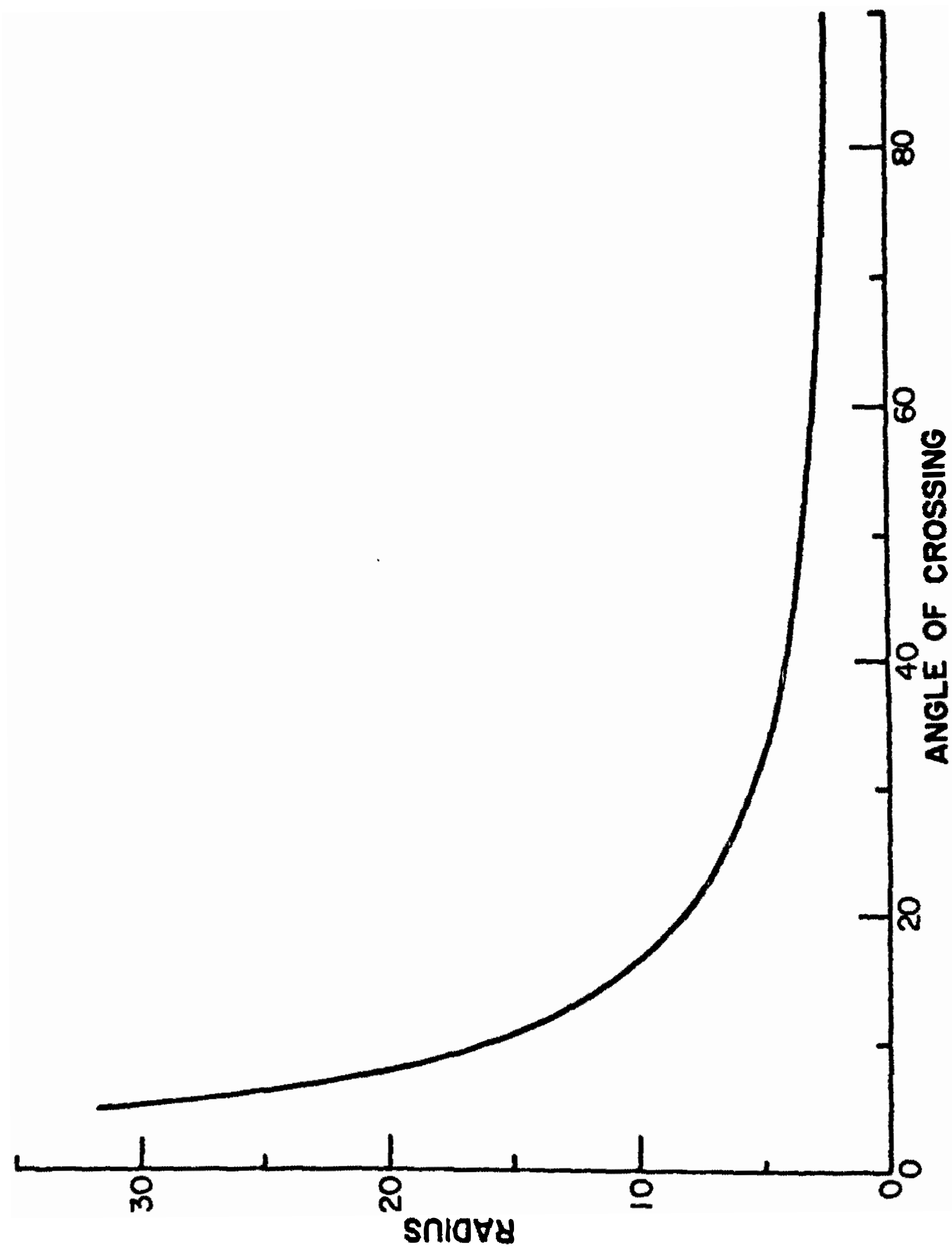

Figure 7 Radius of the 95% Confidence Circle when $\sigma_1 = \sigma_2 = 1$ and $\rho_{12} = 0$

4.4 Example 4

Repeat Example 3 with correlation $\rho_{12} = 0.5$. Algorithm 1 with n = 3 gives σ_x = 36.1325 m, σ_y = 9.3864 m, θ = 19.5924°, and p(30 m) = 0.5666 with $E = 1.9*10^{-8}$.

Algorithm 2 with n = 6 gives for the 95% circle, R = 71.4658 m and $E = 5.8*10^{-8}$. The radius of the 99.9% circle is calculated with n = 7 to be R = 119.2786 m where $E = 6.1*10^{-7}$.

The 95% ellipse has parameters $k\sigma_x$ = 88.4433 m, $k\sigma_y$ = 22.9756 m, and θ = 19.5924°. The area of the 95% circle, 16045.2 m^2 is 151% greater than the area of the 95% confidence ellipse, 6383.8 m^2.

Comparing the results of Examples 3 and 4, it may be observed that the effect of changing the correlation from zero to 0.5 is to increase by 41% the area of the 95% circle while the area of the 95% ellipse is decreased by 13%. Moreover, the orientation of the 95% ellipse is increased from 15.7733° to 19.5924°.

These examples suggest that confidence ellipses are superior to confidence circles since they provide the same probability of location but over a significantly smaller area. To be more precise, for any legitimate values of σ_1, σ_2, α, and ρ_{12}, the area of the 95% ellipse is $\Pi*\ln(400)\sigma_x\sigma_y$ while the area of the 95% circle is less than the area of the 2dRMS circle, $4\Pi(\sigma_x^2 + \sigma_y^2)$.

In the best of circumstances, that is when $\sigma_1 = \sigma_2$, $\alpha = \Pi/2$, and $\rho_{12} = 0$, the area of the 95% circle is equal to the area of the 95% ellipse. However, as Example 1 shows, in less than ideal conditions the 95% circle can be several hundred percent larger than the 95% ellipse. Clearly, in such situations, for any probability the confidence ellipse is to be preferred over the confidence circle since a substantially smaller area provides the same probability of location.

5.0 EQUIVALENT FORMULAS FOR THE ERROR ELLIPSE

Defining

$$A = \sigma_1^2 + 2\rho_{12}\sigma_1\sigma_2\cos(\alpha) + \sigma_2^2$$

$$B = 2[1 - \rho_{12}^2]^{\frac{1}{2}}\sigma_1\sigma_2\sin(\alpha)$$

$$C = \sigma_1^2\cot(\alpha) + \rho_{12}\sigma_1\sigma_2\csc(\alpha)$$

it can be shown that the semimajor and semiminor axes of the error ellipse may be calculated from

$$\sigma_x^2 = \frac{1}{2}\csc^2(\alpha)[A + [A^2 - B^2]^{\frac{1}{2}}]$$

$$\sigma_y^2 = \frac{1}{2}\csc^2(\alpha)[A - [A^2 - B^2]^{\frac{1}{2}}]$$

or

$$\sigma_x^2 = \frac{1}{2} A*\csc^2(\alpha) + C*\csc(2\theta)$$

$$\sigma_y^2 = \frac{1}{2} A*\csc^2(\alpha) - C*\csc(2\theta).$$

5.1 Special Cases

For the special case $\sigma_1 = \sigma_2 = \sigma$, and $\rho_{12} = 0$, it can be shown that the parameters of the error ellipse simplify to the following:

$$\sigma_x = \sigma * \csc(\alpha)[1 + |\cos(\alpha)|]^{\frac{1}{2}}$$

$$\sigma_y = \sigma * \csc(\alpha)[1 - |\cos(\alpha)|]^{\frac{1}{2}}$$

$$\theta = \alpha/2.$$

Burt, Kaplan, and Keenly's [4] and Bowditch's [2] formulas for this special case must be used with caution since their formulas for σ_x and σ_y implicitly require that the crossing angle between the two LOPs must be acute. Their formulas give incorrect results for obtuse crossing angles.

Now if $\sigma_1 = \sigma_2 = \sigma$, $\rho_{12} = 0$, and α is restricted to values strictly between 0 and $\pi/2$, then σ_x and σ_y may be further simplified to

$$\sigma_x = 2^{-\frac{1}{2}} \sigma * \csc(\alpha/2)$$

$$\sigma_y = 2^{-\frac{1}{2}} \sigma * \sec(\alpha/2)$$

Finally, if $\sigma_1 = \sigma_2 = \sigma$, $\rho_{12} = 0$, and $\alpha = \pi/2$, then all calculations can be greatly simplified to the circular normal distribution:

$$\sigma_x = \sigma$$
$$\sigma_y = \sigma$$
$$\theta = 0$$

$$p(R) = 1 - e^{-\frac{1}{2}\left[\frac{R}{\sigma}\right]^2}$$

$$R(p) = \sigma\left[-2*\ln(1 - p)\right]^{\frac{1}{2}}.$$

6.0 APPLICATION TO LORAN-C

Bregstone [3], Collins [5], Pierce, McKenzie, and Woodward [8], and Worrell [12] state explicitly or assume implicitly that assumptions (1), (2), and (3) listed in Section 2.2 with $\rho_{12} = 0$ may be applied to LORAN-C. Swanson [9] also accepts the three assumptions but suggests a value of 0.5 for the correlation of the time-difference or TD errors.

Amos and Feldman [1] point out that the TD error is a function of many variables. In reality, because of the current design of many LORAN-C receivers, the central limit theorem of probability theory applies and it is reasonable to assume that the TD errors are approximately normally distributed.

The value for the correlation ρ_{12} is often taken as zero; however, it is likely that another value such as 0.5 should be used. Significant differences in the sizes and orientations of confidence ellipses as well as the sizes of confidence circles may be observed if the correlation is taken as 0.5 instead of zero.

The U.S. Coast Guard periodically publishes revised specifications of the transmitted LORAN-C signal. In this respect, see reference [11]. The current value given for the standard deviation of the TD erors is 100 nanoseconds.

7.0 CONCLUSIONS AND RECOMMENDATIONS

Algorithms with new stopping criteria have been given which may be used to solve two standard problems in position location: (1) Find the probability p that the true position T is within a circle of radius R centered at the observed position O; and, (2) Find the radius R of the circle C centered at O such that the probability is p that T lies within C.

It is assumed that the errors associated with the lines of position may be approximated by a nonorthogonal bivariate dependent Gaussian distribution where the errors are measured orthogonally to the LOPs. The algorithms presented for this model are readily implemented on a microcomputer. Moreover, they are practical since they avoid the use of probability curves, tables, charts, nomograms, fictitious functions and angles of cut, special ratios, sigma star factors, double Langrangian interpolation, and Bessel functions which are required by some methods of solution.

Numerical results confirm the high accuracy and efficiency of the algorithms presented herein for the calculation of the parameters associated with the error ellipse and confidence circles.

Confidence circles are conceptually easily understood and frequently used; however, with the advent of microcomputers with powerful graphics capabilities, confidence ellipses should be considered as a superior alternative in applications where confidence circles have traditionally been used since much less computation is required for the parameters of a confidence ellipse than for a confidence circle. Moreover, the area of a confidence ellipse is generally substantially less than the area of a confidence circle having the same associated probability; this can be important not only in routine position location, but even more so, in critical search and rescue missions.

Finally, as previously stated, the algorithms are appropriate only when the error model described in Section 2.2 is valid for the particular position location system under consideration. Also note that the algorithms must be modified in situations such as the missile or target problem where the errors are measured parallel to the axes of a coordinate system rather than orthogonally to the LOPs as is the case in position location calculations.

APPENDIX: CIRCULAR ERROR PROBABILITIES

K/c	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	.07965567	.04439882	.02421197	.01641775	.01238758	.00993781	.00829412	.00711578	.00623002	.00554007	.00498752
0.2	.15851942	.13397841	.08845339	.06283969	.04824140	.03901935	.03271241	.02814165	.02468247	.02197579	.01980133
0.3	.23582284	.22138043	.17393007	.13182815	.10391932	.08515354	.07191031	.06213865	.05465986	.04876397	.04400252
0.4	.31084348	.30102290	.26351819	.21390853	.17420456	.14518087	.12379829	.10762379	.09504961	.08503269	.07688365
0.5	.38292492	.37558843	.34817902	.30030019	.25329533	.21528872	.18574489	.16268303	.14439413	.12962866	.11750310
0.6	.45149376	.44577086	.42556056	.38463741	.33573842	.29146823	.25481781	.22511143	.20097981	.18117832	.16472979
0.7	.51607270	.51150481	.49606835	.46332588	.41708629	.36993053	.32803032	.29256543	.26293740	.23815834	.21729546
0.8	.57628920	.57259569	.56044571	.53493877	.49418829	.44742080	.40256292	.36271224	.32834532	.29897012	.27385096
0.9	.63187975	.62887213	.61913541	.59931400	.56515643	.52139984	.47593757	.43336291	.39532797	.36201358	.33302319
1.0	.68268949	.68023254	.67235867	.65682424	.62912495	.59009533	.54613196	.50257901	.46214212	.42575533	.39346934
1.1	.72866788	.72665967	.72026823	.70796818	.68593673	.65244889	.61163161	.56874674	.52724621	.48878740	.45392557
1.2	.76986066	.76822148	.76303049	.75321755	.73585580	.70799732	.67142689	.63061681	.58934943	.54987365	.51324774
1.3	.80639903	.80506480	.80085535	.79299679	.77935506	.75672656	.72496735	.68731223	.64743948	.60798223	.57044264
1.4	.83848668	.83740489	.83400178	.82770477	.81698517	.79892884	.77208895	.73830894	.70078999	.66230358	.62468890
1.5	.86638560	.86551266	.86277282	.85773618	.84930716	.83508160	.81292873	.78339628	.74895002	.71225465	.67534753
1.6	.89040142	.88970083	.88750602	.88349137	.87686446	.86575592	.84783930	.82262457	.79171937	.75747088	.72196270
1.7	.91086907	.91031019	.90856194	.90537663	.90017456	.89155362	.87731164	.85624712	.82911370	.79778816	.76425392
1.8	.92813936	.92769639	.92631248	.92379894	.91972753	.91306800	.90191102	.88466237	.86132384	.83321750	.80210130
1.9	.94256688	.94221819	.94112996	.93915857	.93598555	.93086154	.92222772	.90836088	.88867314	.86391495	.83552554
2.0	.95449974	.95422722	.95337750	.95184149	.94938155	.94545458	.93884177	.92787988	.91157619	.89014951	.86466472
2.1	.96427116	.96405976	.96340112	.96221269	.96031702	.95732052	.95229986	.94376684	.93050133	.91227137	.88974947
2.2	.97219310	.97203038	.97152372	.97061093	.96915971	.96688448	.96310169	.95655220	.94593857	.93068211	.91107838
2.3	.97855178	.97842751	.97804079	.97734503	.97624187	.97452393	.97169345	.96673063	.95837388	.94580848	.92899465
2.4	.98360493	.98351079	.98321798	.98269178	.98185941	.98057026	.97846612	.97474955	.96826981	.95808039	.94386524
2.5	.98758067	.98750994	.98729005	.98689528	.98627204	.98531115	.98375690	.98100352	.97605221	.96791357	.95606307
2.6	.99067762	.99062493	.99046116	.99016742	.98970446	.98899336	.98785268	.98583311	.98210228	.97569685	.96595255
2.7	.99306605	.99302712	.99290619	.99268943	.99234833	.99182603	.99099441	.98952681	.98675296	.98178371	.97387859
2.8	.99488974	.99486123	.99477268	.99461409	.99436485	.99398423	.99338209	.99232491	.99028880	.98648759	.98015891
2.9	.99626837	.99624767	.99618340	.99606837	.99588778	.99561263	.99517978	.99442459	.99294821	.99008026	.98507921
3.0	.99730020	.99728531	.99723907	.99715634	.99702662	.99682936	.99652052	.99598541	.99492739	.99279253	.98889100
3.1	.99806479	.99805417	.99802119	.99796223	.99786985	.99772961	.99751096	.99713480	.99638509	.99481678	.99181130
3.2	.99862572	.99861821	.99859490	.99855325	.99848804	.99838920	.99823562	.99797327	.99744776	.99631047	.99402398
3.3	.99903315	.99902789	.99901156	.99898239	.99893677	.99886771	.99876073	.99857919	.99821466	.99740035	.99568216
3.4	.99932614	.99932249	.99931115	.99929092	.99925928	.99921145	.99913755	.99901292	.99876261	.99818678	.99691128
3.5	.99953474	.99953223	.99952443	.99951052	.99948877	.99945594	.99940533	.99932046	.99915025	.99874802	.99781251
3.6	.99968178	.99968007	.99967476	.99966527	.99965047	.99962813	.99959377	.99953644	.99942181	.99914419	.99846619
3.7	.99978440	.99978324	.99977965	.99977325	.99976326	.99974820	.99972508	.99968668	.99961019	.99942084	.99893523
3.8	.99985530	.99985453	.99985213	.99984785	.99984117	.99983111	.99981568	.99979017	.99973960	.99961195	.99926820
3.9	.99990381	.99990329	.99990170	.99989886	.99989444	.99988778	.99987758	.99986078	.99982765	.99974257	.99950204
4.0	.99993666	.99993632	.99993527	.99993341	.99993051	.99992614	.99991946	.99990849	.99988697	.99983090	.99966454
4.1	.99995868	.99995847	.99995779	.99995657	.99995468	.99995185	.99994751	.99994041	.99992656	.99989002	.99977625
4.2	.99997331	.99997317	.99997273	.99997195	.99997073	.99996890	.99996611	.99996156	.99995273	.99992917	.99985225
4.3	.99998292	.99998283	.99998255	.99998205	.99998127	.99998011	.99997833	.99997544	.99996985	.99995483	.99990341
4.4	.99998917	.99998912	.99998894	.99998863	.99998813	.99998740	.99998628	.99998445	.99998095	.99997147	.99993748
4.5	.99999320	.99999317	.99999306	.99999286	.99999255	.99999209	.99999139	.99999025	.99998808	.99998216	.99995993
4.6	.99999578	.99999575	.99999568	.99999556	.99999537	.99999508	.99999465	.99999395	.99999261	.99998895	.99997458
4.7	.99999740	.99999738	.99999734	.99999727	.99999715	.99999697	.99999671	.99999628	.99999546	.99999322	.99998403
4.8	.99999841	.99999841	.99999838	.99999833	.99999826	.99999816	.99999799	.99999773	.99999724	.99999588	.99999007
4.9	.99999904	.99999904	.99999902	.99999899	.99999895	.99999889	.99999879	.99999863	.99999833	.99999752	.99999389
5.0	.99999943	.99999942	.99999941	.99999940	.99999937	.99999933	.99999928	.99999918	.99999901	.99999852	.99999627
5.1	.99999966	.99999966	.99999965	.99999964	.99999963	.99999961	.99999957	.99999952	.99999941	.99999913	.99999775
5.2	.99999980	.99999980	.99999980	.99999979	.99999978	.99999977	.99999975	.99999972	.99999966	.99999949	.99999866
5.3	.99999988	.99999988	.99999988	.99999988	.99999987	.99999987	.99999985	.99999984	.99999980	.99999971	.99999921
5.4	.99999993	.99999993	.99999993	.99999993	.99999993	.99999992	.99999992	.99999991	.99999989	.99999983	.99999953
5.5	.99999996	.99999996	.99999996	.99999996	.99999996	.99999996	.99999995	.99999995	.99999993	.99999990	.99999973
5.6	.99999998	.99999998	.99999998	.99999998	.99999998	.99999998	.99999997	.99999997	.99999996	.99999995	.99999985
5.7	.99999999	.99999999	.99999999	.99999999	.99999999	.99999999	.99999998	.99999998	.99999998	.99999997	.99999991
5.8	.99999999	.99999999	.99999999	.99999999	.99999999	.99999999	.99999999	.99999999	.99999999	.99999998	.99999995
5.9	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	.99999999	.99999999	.99999999	.99999997
6.0								1.00000000	1.00000000	1.00000000	.99999998
6.1											.99999999
6.2											1.00000000

p(K,c) = probability that a point lies within a circle whose center is at the origin and whose radius is $R = K\sigma_x$. Here $c = \sigma_y/\sigma_x$ where σ_x is the larger standard deviation. The table gives values of the standard orthogonal bivariate independent Gaussian distribution.

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