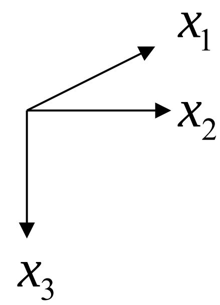
VI. Theoretical Fundamentals of Airborne Gradiometry

- Gravity Gradients Review
- Why Gradiometry?
- Gradiometry equation
- Instrumentation & Existing Systems (non-inclusive)
- Rudimentary Error Analysis

Gravitational Quantities in Cartesian Coordinates

local coordinate frame



- **gravitational potential:** $V(x_1, x_2, x_3)$ **(0-order tensor)**

- **gravitation vector:** $\boldsymbol{g} = \nabla V = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial V}{\partial x_1} \\ \dfrac{\partial V}{\partial x_2} \\ \dfrac{\partial V}{\partial x_3} \end{pmatrix}$ **(1st-order tensor)**

- **gravitational gradient tensor:** $\Gamma = \nabla \nabla^{\mathrm{T}} V = \nabla \boldsymbol{g}^{\mathrm{T}}$ **(2nd-order tensor)**

Gravitational Gradient Tensor

$$\boldsymbol{\Gamma} = \nabla \nabla^{\mathrm{T}} V = \nabla \boldsymbol{g}^{\mathrm{T}} = \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{pmatrix} \begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_3}{\partial x_1} \\ \dfrac{\partial g_1}{\partial x_2} & \dfrac{\partial g_2}{\partial x_2} & \dfrac{\partial g_3}{\partial x_2} \\ \dfrac{\partial g_1}{\partial x_3} & \dfrac{\partial g_2}{\partial x_3} & \dfrac{\partial g_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{23} & \Gamma_{33} \end{pmatrix}$$

- **In-line gradients:** $\Gamma_{jj} = \dfrac{\partial^2 V}{\partial x_j^2} = \dfrac{\partial g_j}{\partial x_j}$
- **Cross gradients:** $\Gamma_{jk} = \dfrac{\partial^2 V}{\partial x_j \partial x_k} = \dfrac{\partial g_k}{\partial x_j}$
- ***V*** **is twice continuously differentiable (continuous density) $\Rightarrow$ Γ is symmetric**

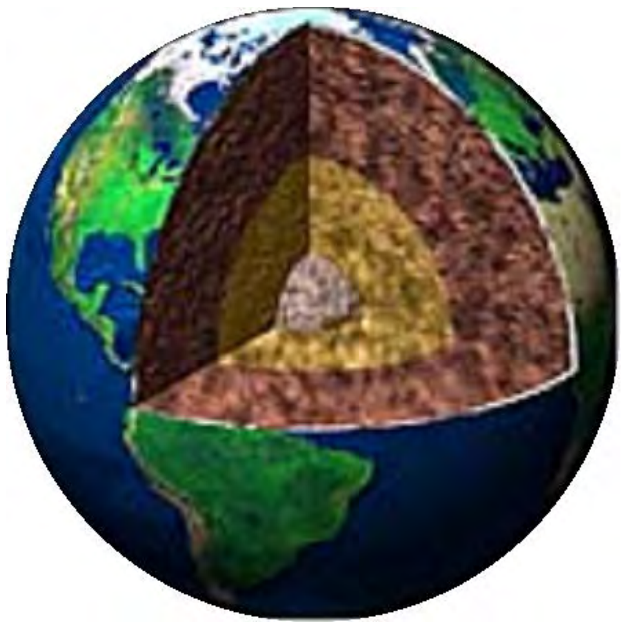
$$\frac{\partial^2 V}{\partial x_k \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_k} \qquad \left(\Gamma_{kj} = \Gamma_{jk} \right)$$

- **Poisson's equation $\Rightarrow$ in-line gradients are linearly dependent; e.g.**

$$\Gamma_{33} = -\Gamma_{11} - \Gamma_{22} - 4\pi G \rho$$

Γ has 5 independent elements

Fractional Contributions by Depth of Source



Earth Radius:
~ 6371 km

- crust: 0 - ~20 km
- upper mantle: ~20 km - 400 km
- transition/lower mantle: 400 km - 2890 km
- outer core: 2890 km - 5150 km
- inner core: 5150 km - 6370 km

disturbing potential → T; gravity disturbance → δg; gradient disturbance → $\delta\Gamma$

depth [km]	T	δg	$\delta\Gamma$
0.15	7%	89%	99.995%
350	44%	9%	0.005%
2880	49%	2%	--

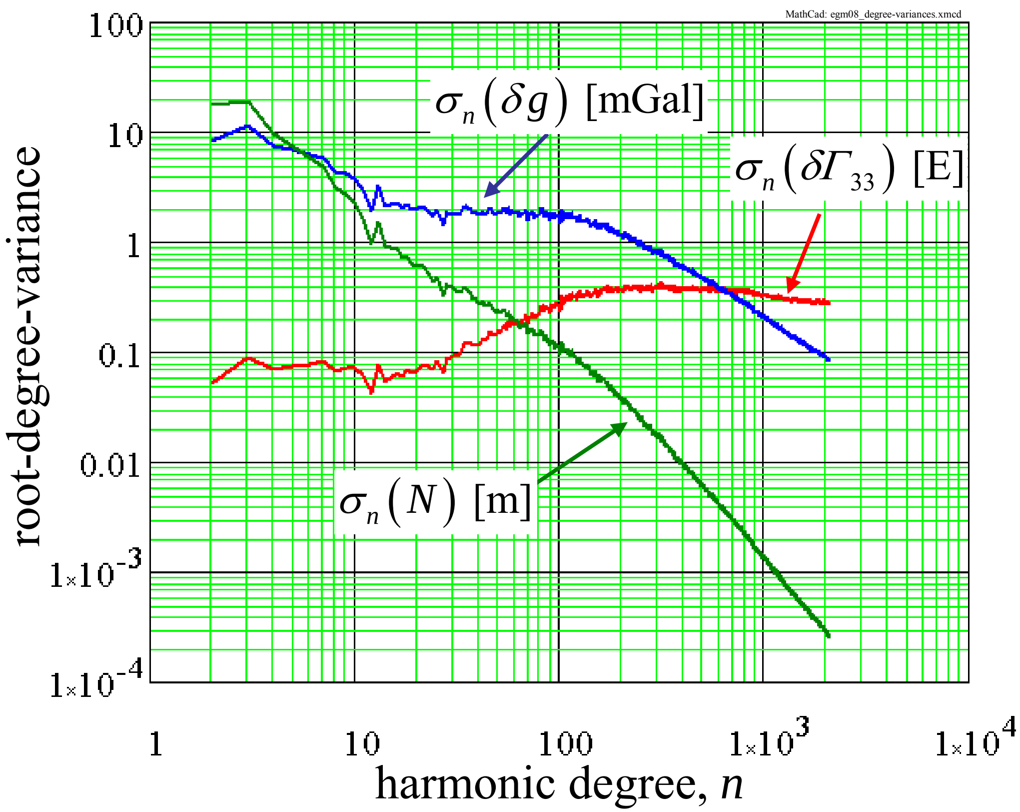
after Jordan (1978)

Global Degree Variances

- **Disturbing potential:** $$T(\theta,\lambda,r) = \frac{GM}{a}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\left(\frac{a}{r}\right)^{n+1} C_{n,m}\bar{Y}_{n,m}(\theta,\lambda)$$

  – **root-degree-variances:**

EGM2008 model



◦ **geoid undulation**

$$\sigma_n(N) = a\sqrt{\sum_{m=-n}^{n} C_{n,m}^2}$$

◦ **gravity disturbance**

$$\sigma_n(\delta g) = \frac{GM}{a^2}(n+1)\sqrt{\sum_{m=-n}^{n} C_{n,m}^2}$$

◦ **radial gravitational gradient**

$$\sigma_n(\delta\Gamma_{33}) = \frac{GM}{a^3}(n+1)(n+2)\sqrt{\sum_{m=-n}^{n} C_{n,m}^2}$$

Upward Attenuation vs. High-Frequency Enhancement

- **Gravitational gradient attenuates as r^{-3}, gravitation as r^{-2}**
  - **upward continuation frequency response:** $e^{-2\pi \bar{f} h}$ $\quad \bar{f} = \sqrt{f_1^2 + f_2^2}$

    $h =$ altitude

- **Gradient is more sensitive to high spatial frequencies**
  - **derivative response:** $\left(2\pi \bar{f}\right)^q$

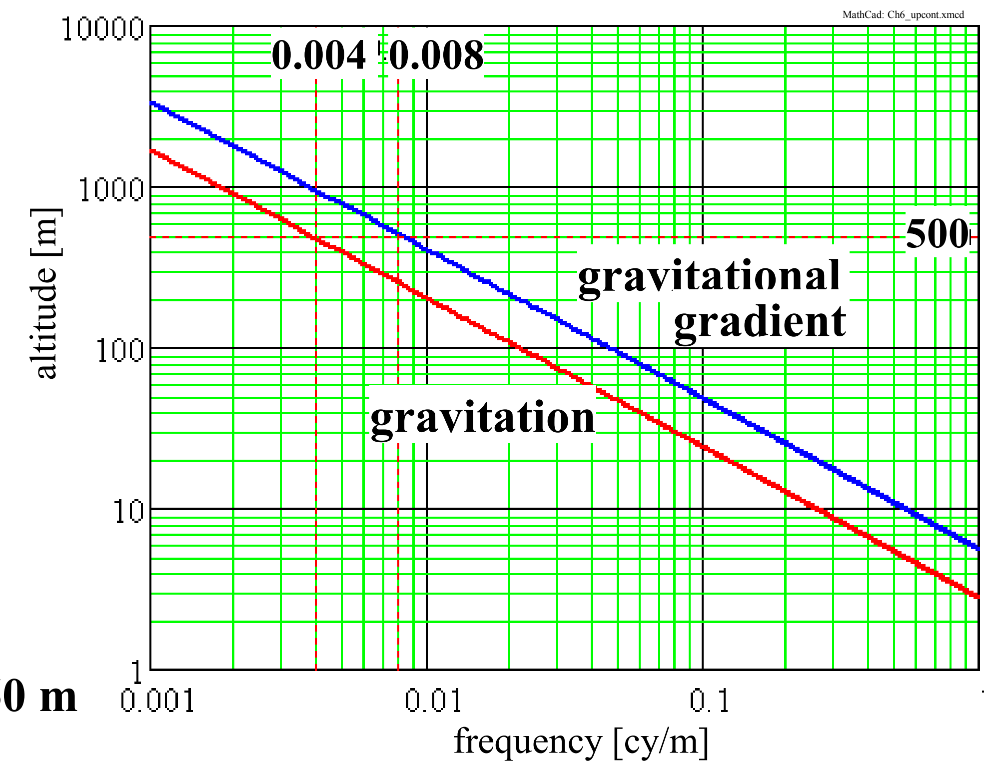
    gravitation $\Rightarrow q = 1$

    gradient $\Rightarrow q = 2$

- **No attenuation at $\bar{f}$, if**

$$R^q \left(2\pi \bar{f}\right)^q e^{-2\pi \bar{f} h} \geq 1$$

R → takes care of units

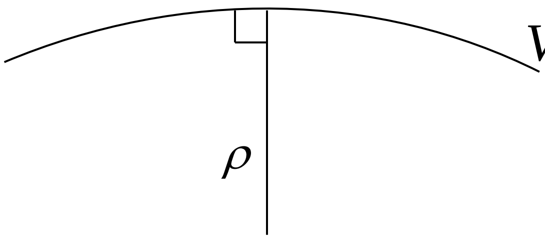


- **For example, at $h = 500$ m**
  - **gravimeter resolves wavelengths > 250 m**
  - **gradiometer resolves wavelengths > 125 m**

Curvature of Equipotential Surface

- **Equipotential surface:** Surface on which $V = V_0$

- **Consider normal section, *s***



$$V(s,z) = V_0$$

parametric equation of $z(s)$
(z is vertical, s is arc length)

- **Formula for curvature of arc, *s* :**

$$\kappa_s = \left|\frac{d^2 z}{ds^2}\right|\left(1 + \frac{dz}{ds}\right)^{-3/2}$$

- **It can be shown* that the curvature is proportional to the gravitational gradient:**

$$\kappa_s = \frac{1}{g}\left|\frac{d^2 V}{ds^2}\right|$$

g is gravitation magnitude

- **Radius of curvature of arc, *s*:** $\rho = \dfrac{1}{\kappa}$

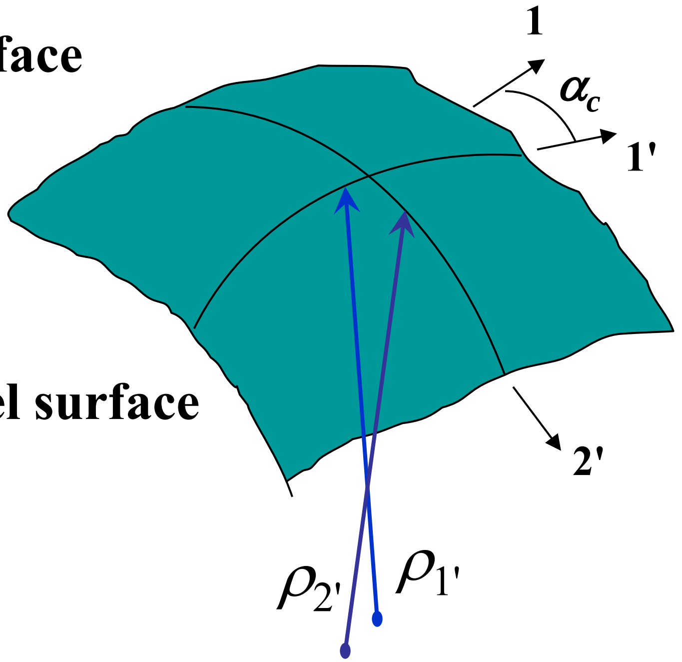
- **The two principal radii of curvature, ρ_1 and ρ_2, represent the minimum and the maximum curvatures (along arcs that are perpendicular to each other)**

* Heiskanen and Moritz, 1967, p.51

Differential Curvature

- **Differential curvature of equipotential (level) surface**

$$\Gamma_C = g_3\left(\frac{1}{\rho_{2'}} - \frac{1}{\rho_{1'}}\right) = \sqrt{\left(2\Gamma_{12}\right)^2 + \left(\Gamma_{22} - \Gamma_{11}\right)^2}$$



- **Direction of minimum curvature**

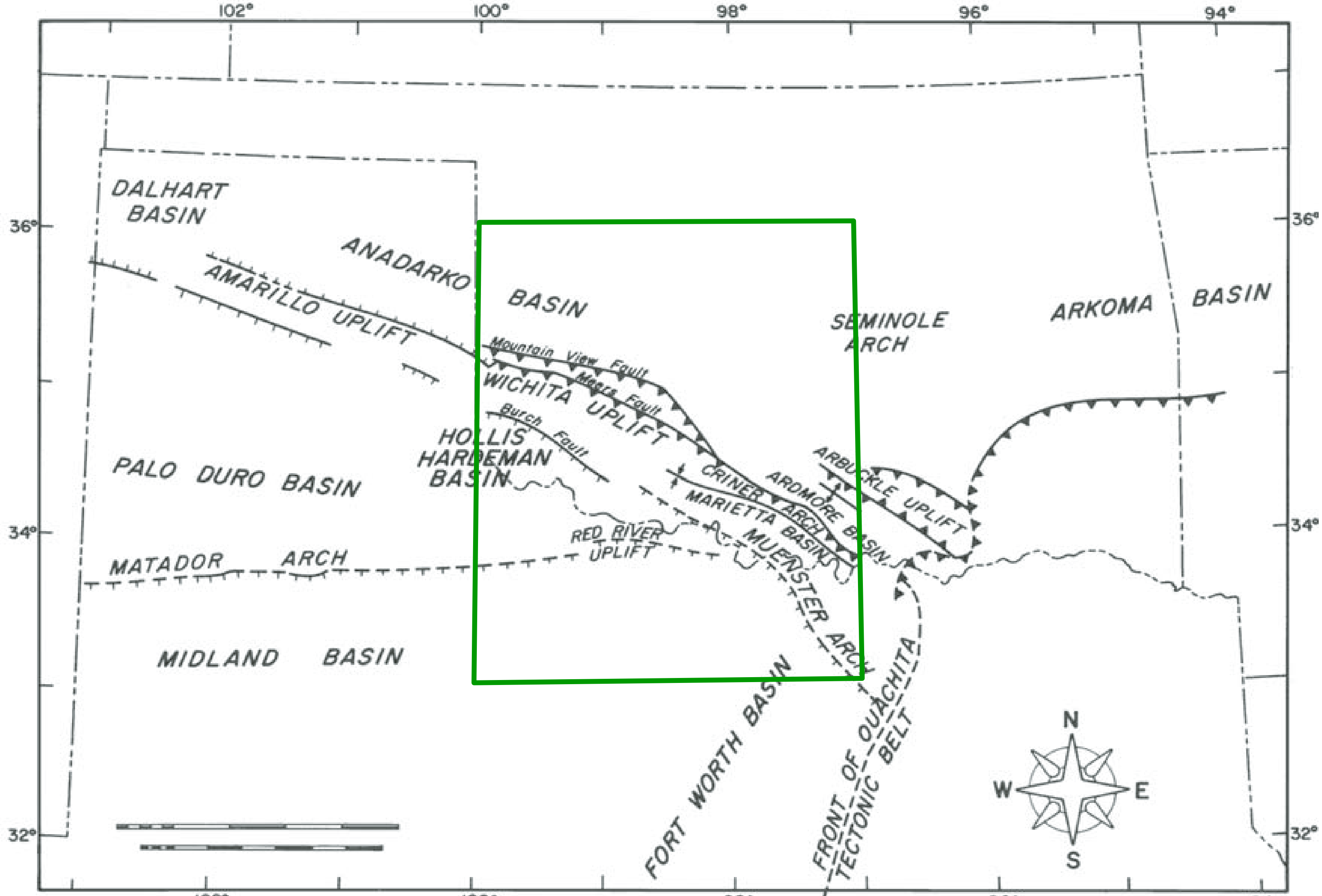
$$\alpha_c = \frac{1}{2}\tan^{-1}\frac{-2\Gamma_{12}}{\Gamma_{22} - \Gamma_{11}}$$

  - **α_c is the azimuth of the direction of minimum curvature**
  - **If $\alpha_c = 0$, coordinate axes, x_1 and x_2, coincide with the directions of minimum and maximum curvature; and $\Gamma_{12} = 0$**

- **Γ_C is particularly suited to map linear features of the mass density structure**

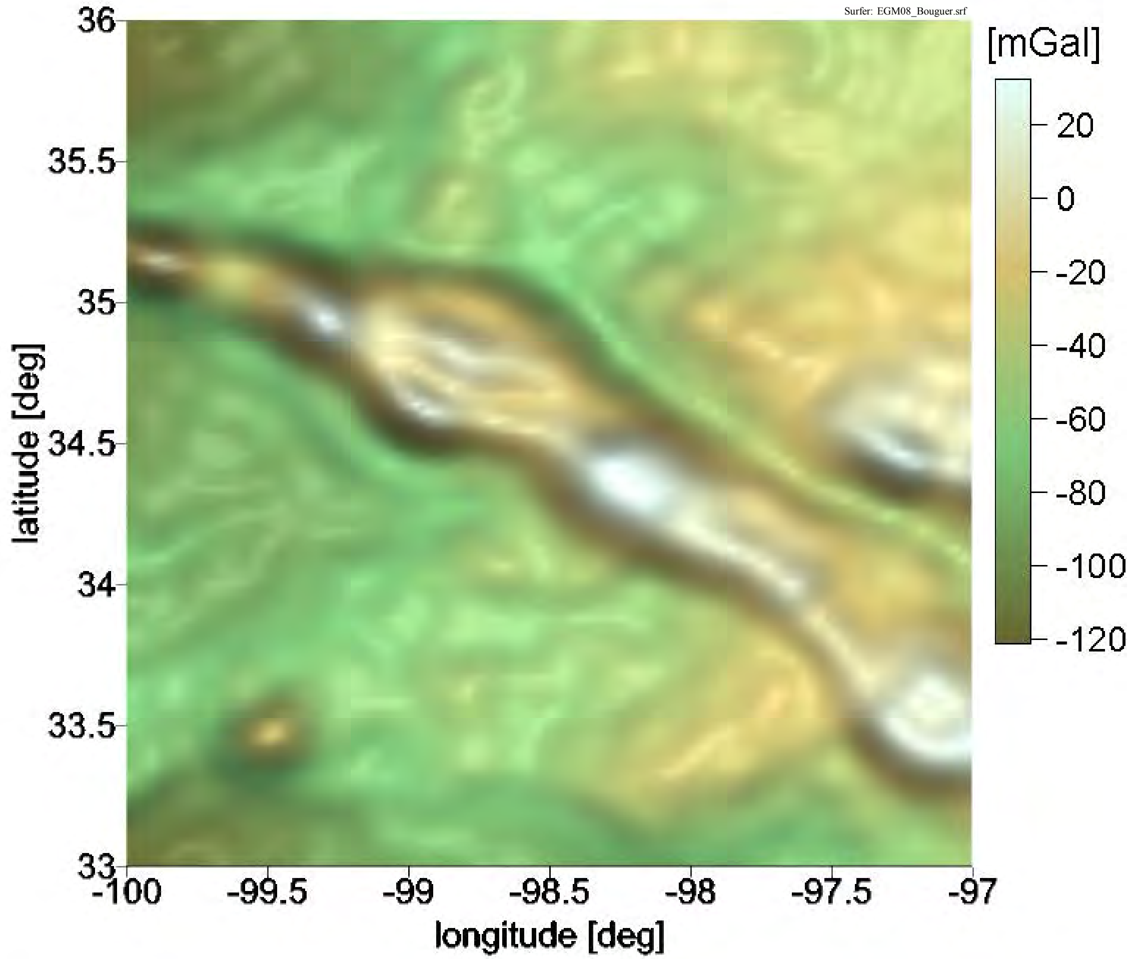
Geography and Major Faults in Wichita Uplift Area

- **Green outline is location of EGM2008 map**



map: [Keller and Baldridge 1995]

EGM2008* Bouguer Anomalies

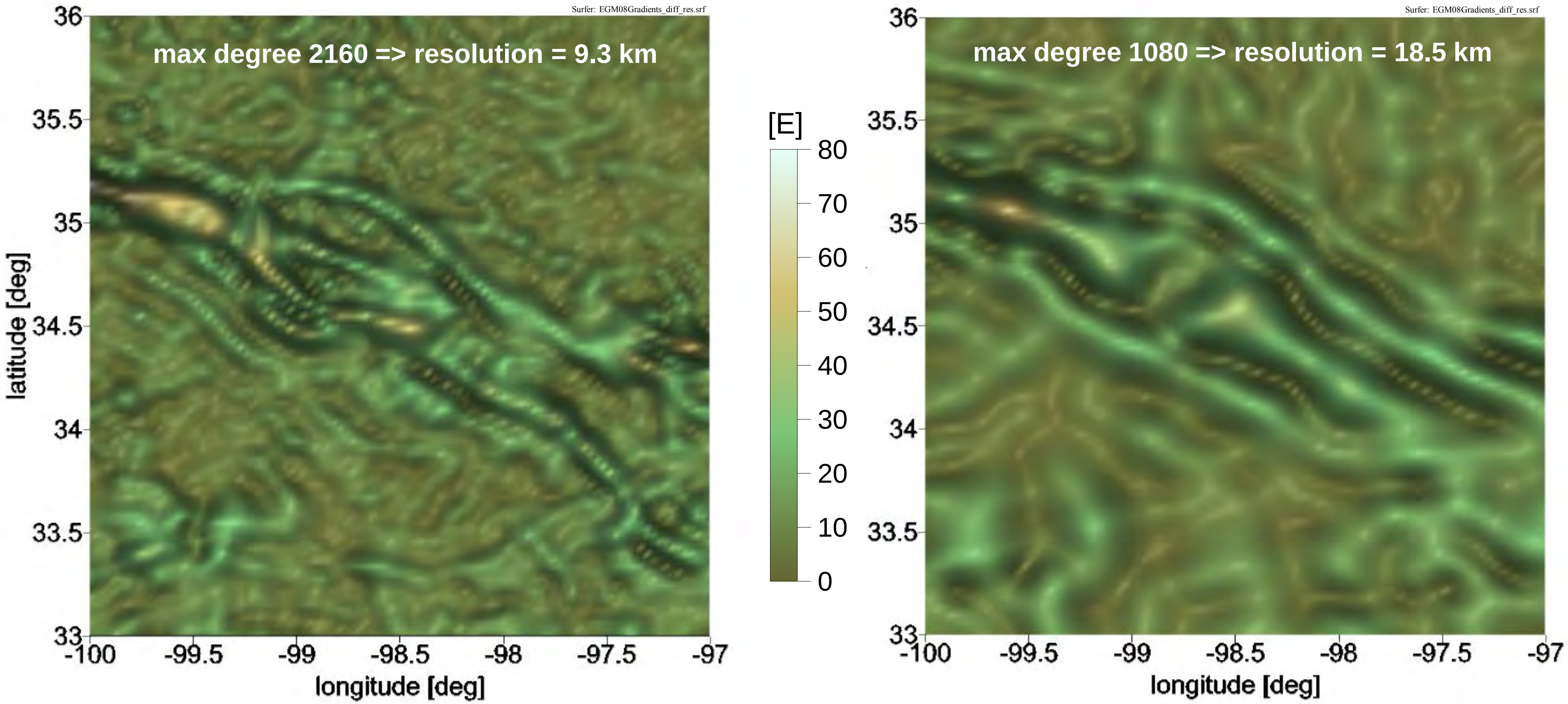


- **Bouguer anomalies roughly indicate major faults**

***Pavlis et al. 2012**

EGM2008 Field Curvature Magnitude

$$\Gamma_C = \sqrt{\left(\Gamma_{22} - \Gamma_{11}\right)^2 + \left(2\Gamma_{12}\right)^2}$$

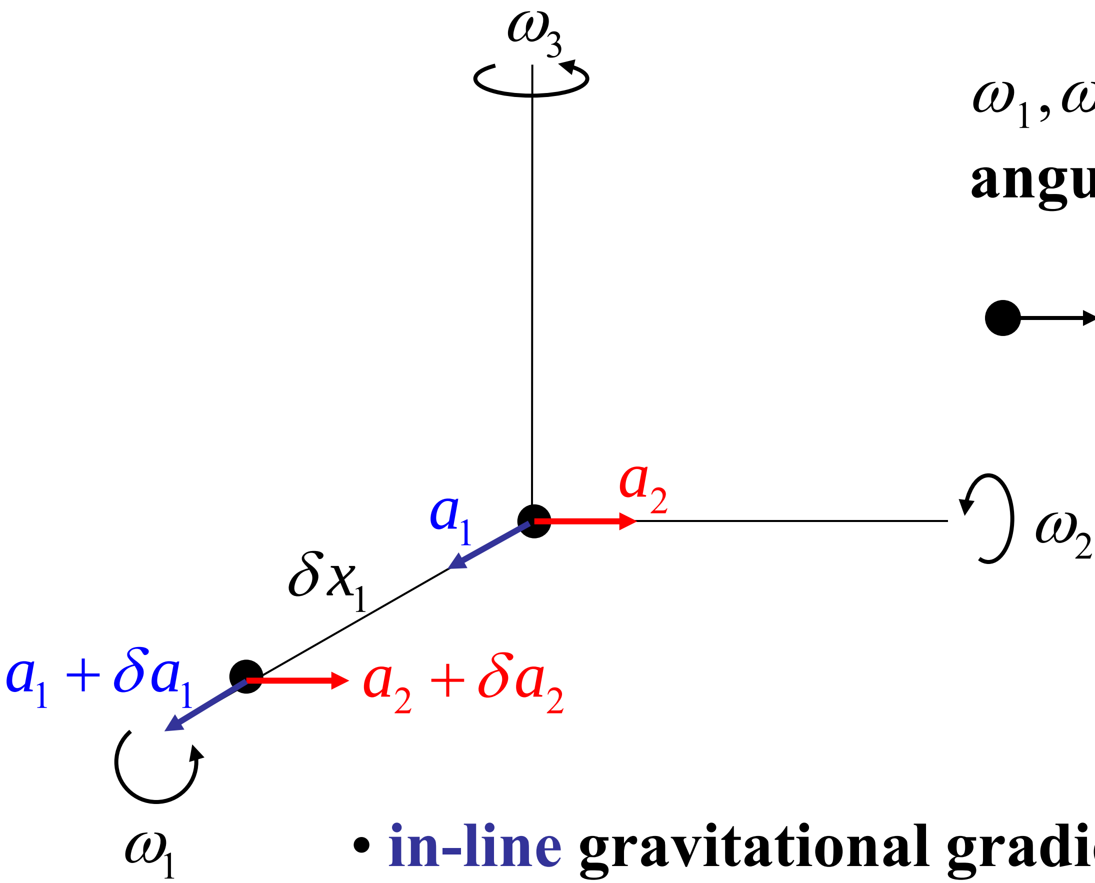


- **Note enhanced details of linear features!**

Moving-Base Gravity Gradiometers – A Brief History

- Torsion balances were replaced after 1938 by highly accurate and rapid-measurement gravimeters for geophysical exploration and geodetic applications
- 1960s through 1980s saw development of gravitational gradiometers specifically for moving-base platforms
  - gradiometers are not sensitive to linear accelerations
  - NASA, U.S. DoD were the main sponsors
- In particular, space-borne gradiometers were proposed for gravitational mapping of planets and moons.
  - a key contender for an Earth-orbiting gravitational mapper was a gradiometer
  - GOCE (2009-2013) was the first (and only) space-borne gradiometer
- Ship-borne and airborne systems developed slowly, in competition with (cheaper) gravimeter/GPS systems
- Successful demonstration of airborne gradiometer in 1980s spurred heavy investment by geophysical exploration companies
  - e.g., Bell Geospace is one of today's leaders

Moving-Base Gradiometer Couples Gravitational Gradient and Angular Rates



$\omega_1, \omega_2, \omega_3$

angular rates about sensor axes

●⟶ **= accelerometer**

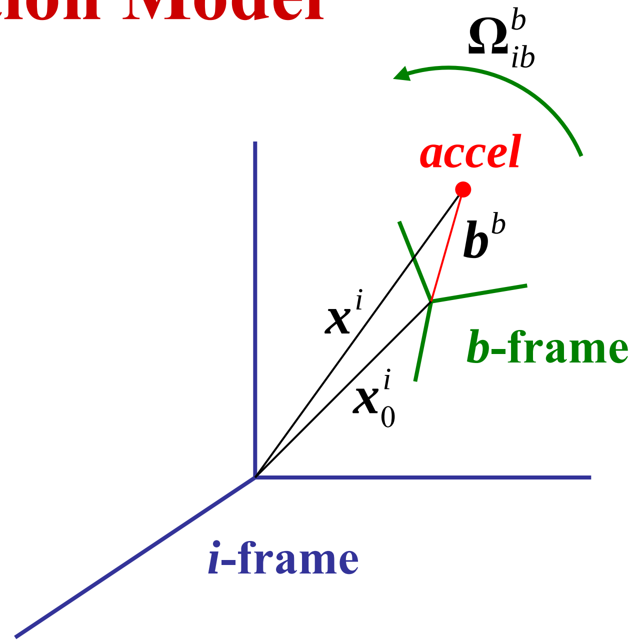
gradiometer measurement

↓

- **in-line gravitational gradient:** $\Gamma_{11} = -\frac{\partial a_1}{\partial x_1} + \omega_2^2 + \omega_3^2$
- **cross gravitational gradient:** $\Gamma_{21} = -\frac{\partial a_2}{\partial x_1} - \omega_1 \omega_2 - \frac{d}{dt}\omega_3$

(equations are derived next slide)

Gravity Gradiometry Observation Model



- Recall **lever-arm equation** for acceleration

$$\ddot{\boldsymbol{x}}^i_{\text{accel}} = \ddot{\boldsymbol{x}}^i_{\text{body}} + \ddot{\boldsymbol{b}}^i$$

– where $\ddot{\boldsymbol{b}}^i = \ddot{C}^i_b \boldsymbol{b}^b = C^i_b \left(\dot{\boldsymbol{\Omega}}^b_{ib} + \boldsymbol{\Omega}^b_{ib} \boldsymbol{\Omega}^b_{ib} \right) \boldsymbol{b}^b$

$$\ddot{\boldsymbol{x}}^i = \boldsymbol{a}^i + \boldsymbol{g}^i$$

- Inertial accelerations in b-frame

$$\boldsymbol{a}^b_{\text{accel}} = \boldsymbol{a}^b_{\text{body}} + \left(\boldsymbol{g}^b_{\text{body}} - \boldsymbol{g}^b_{\text{accel}} \right) + \dot{\boldsymbol{\Omega}}^b_{ib} \boldsymbol{b}^b + \boldsymbol{\Omega}^b_{ib} \boldsymbol{\Omega}^b_{ib} \boldsymbol{b}^b$$

- **Differentiate** with respect to $\boldsymbol{b}^b$

$$\frac{\partial \boldsymbol{a}^b_{\text{accel}}}{\partial \boldsymbol{b}^b} = -\frac{\partial \boldsymbol{g}^b_{\text{accel}}}{\partial \boldsymbol{b}^b} + \dot{\boldsymbol{\Omega}}^b_{ib} + \boldsymbol{\Omega}^b_{ib} \boldsymbol{\Omega}^b_{ib}$$

acceleration gradients in b-frame

gravitational gradients in b-frame

Methods to Isolate Gravitational Gradients

- **General model in the body frame:**

$$\begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_2}{\partial x_1} & \frac{\partial a_3}{\partial x_1} \\ \frac{\partial a_1}{\partial x_2} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_3}{\partial x_2} \\ \frac{\partial a_1}{\partial x_3} & \frac{\partial a_2}{\partial x_3} & \frac{\partial a_3}{\partial x_3} \end{pmatrix} = -\begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{pmatrix} + \begin{pmatrix} 0 & -\dot{\omega}_3 & \dot{\omega}_2 \\ \dot{\omega}_3 & 0 & -\dot{\omega}_1 \\ -\dot{\omega}_2 & \dot{\omega}_1 & 0 \end{pmatrix} - \begin{pmatrix} \omega_2^2 + \omega_3^2 & -\omega_1\omega_2 & -\omega_1\omega_3 \\ -\omega_2\omega_1 & \omega_3^2 + \omega_1^2 & -\omega_2\omega_3 \\ -\omega_3\omega_1 & -\omega_3\omega_2 & \omega_1^2 + \omega_2^2 \end{pmatrix}$$

- **Method 1: use accurate gyros to measure the angular rates, ω_j , or to stabilize the gradiometer platform**
  - **critical for partial tensor gradiometers and FTGs that are not true full tensor gradiometers.**
- **Method 2: use the skew-symmetry and symmetry of the right-hand side matrices:**
  - **for** $j,k,\ell = 1,2,3$

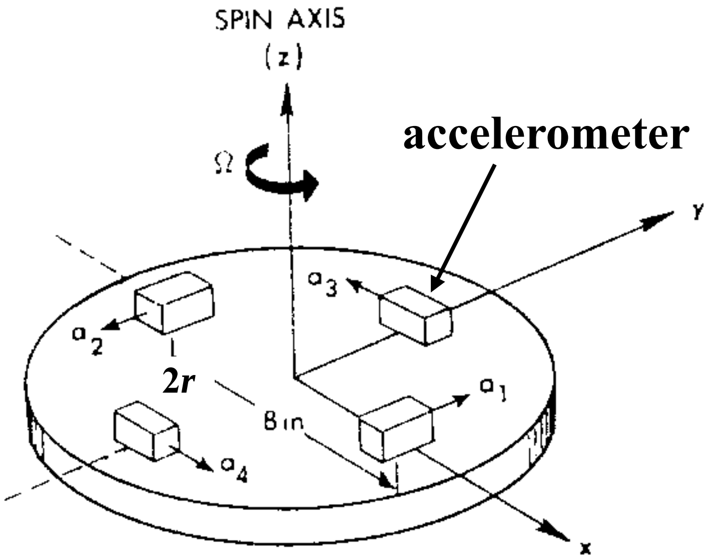
$$\frac{1}{2}\left(\frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j}\right) = \frac{d}{dt}\omega_\ell \;\rightarrow\; \int \;\rightarrow\; \omega_\ell$$

initial ω_ℓ

$$\omega_j\omega_k - \frac{1}{2}\left(\frac{\partial a_k}{\partial x_j} + \frac{\partial a_j}{\partial x_k}\right) = \Gamma_{jk}$$

$$-\omega_k^2 - \omega_\ell^2 - \frac{\partial a_j}{\partial x_j} = \Gamma_{jj}$$

Bell Aerospace / Lockheed Martin Gradiometer



Gravity Gradiometer Instrument (GGI) rotating sensor disk

- **Data are differential curvature components**

  $\Gamma_{22} - \Gamma_{11} \qquad \Gamma_{12}$

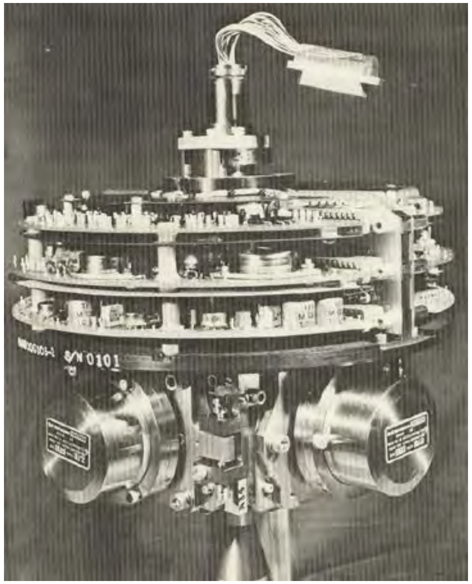
- **They are modulated at twice the rotation rate, Ω, of the sensor disk.**
  - many errors modulate at once the rotation rate and can be eliminated in the demodulation of the output

$$a_1(t) + a_2(t) - \left(a_3(t) + a_4(t)\right) = 2r\left(\Gamma_{22} - \Gamma_{11}\right)\sin 2\Omega t + 4r\Gamma_{12}\cos 2\Omega t$$

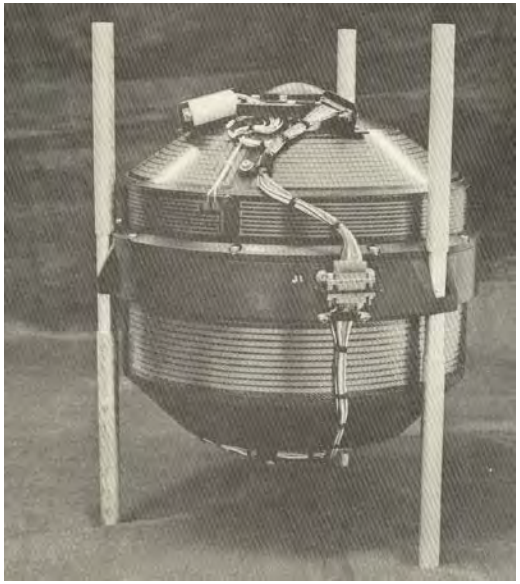
- **Four-accelerometer version was used for DMA/Air Force Gravity Gradiometer Survey System (GGSS) in 1980s**

- **Also used by Navy and still by Bell Geospace in their FTG system**

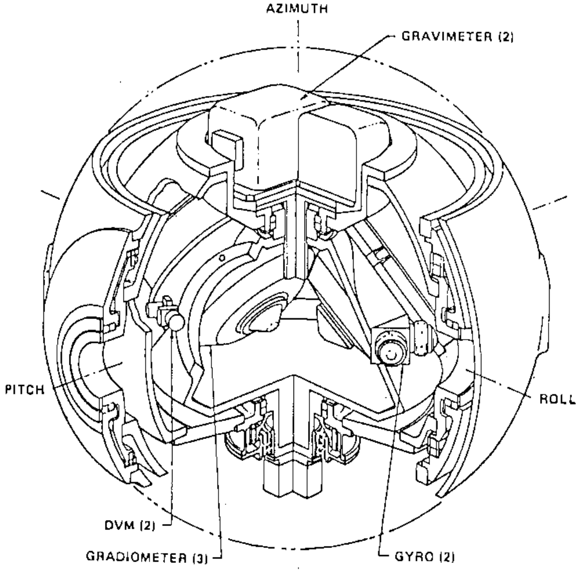
Bell Aerospace GGSS



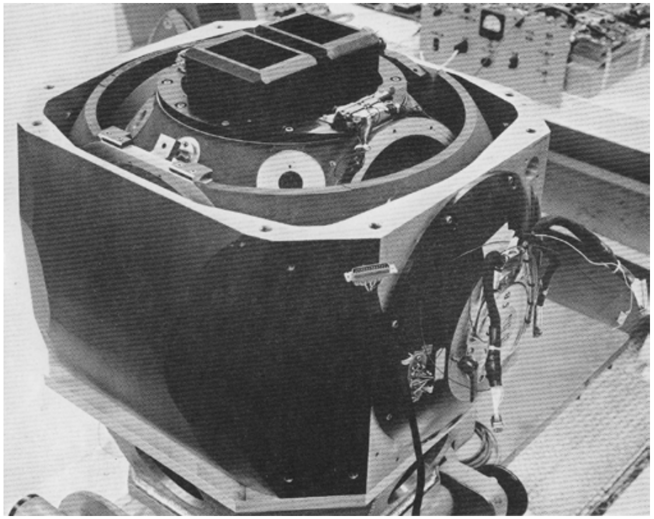
GGI disk and electronics



GGI unit



“Umbrella” configuration of three (3) GGIs on a stabilized platform



Gravity Gradiometer Survey System (GGSS)

all pictures: Bell Aerospace GGSS Proposal

Bell Aerospace GGSS (continued)

- Designed for ground-vehicle and airborne deployment

(Jekeli 1988)

- Tests in 1980's demonstrated first airborne gradiometer system
- Accuracy of about 3 mgal in vertical gravity disturbance, 4-6 mgal in horizontal components in airborne case (Jekeli 1993)

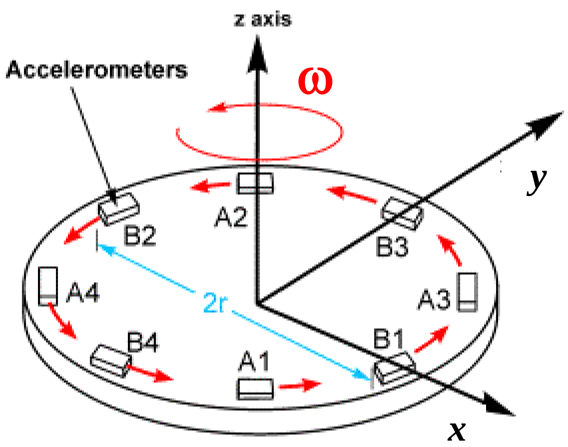
Bell Aerospace GGSS (continued)

- **Calibration for self-gradients: gradients due to near-field mass of aircraft**
  - **these are not constant since gradiometer system is mounted on a stabilized platform, while the aircraft pitches and rolls**
  - **determine sensor output as airplane rolls and pitches statically (no changing Earth gravitational gradient sources)**

(Jekeli 1988)

Further Developments by Lockheed Martin

- **Two sets of 4 accelerometers on the rotating platform; one rotated 45° from the other**



accelerometer platform of Gravity Gradiometer Instrument (GGI)

  – **add and subtract accelerometer outputs in sets A and B:**

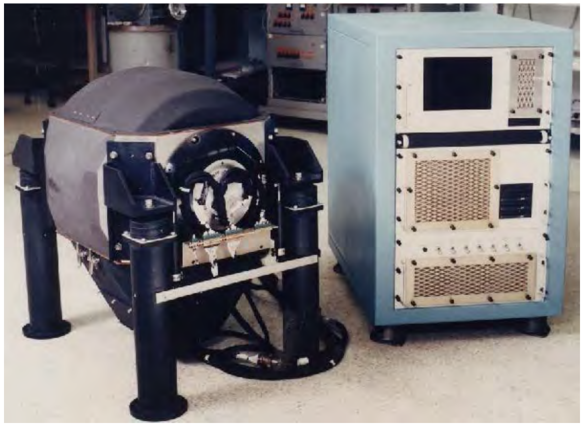
$$A1(t)+A2(t)-\left(A3(t)+A4(t)\right)=$$
$$2r\left(\Gamma_{22}-\Gamma_{11}\right)\sin 2\omega t+4r\Gamma_{12}\cos 2\omega t$$

$$B1(t)+B2(t)-\left(B3(t)+B4(t)\right)=$$
$$2r\left(\Gamma_{22}-\Gamma_{11}\right)\cos 2\omega t-4r\Gamma_{12}\sin 2\omega t$$

- **Single-axis instrument yields two curvature components**
  – **2 x accelerometers => doubles the signal amplitude (both A and B sets yield the curvature components upon demodulation)**
- **Full-tensor gradiometer still needs at least 3 such GGIs**

Airborne Gradiometry - BHP Falcon™ (Fugro …)

Instrumentation



- **Airborne Gravity Gradiometer (AGG)**
  - **Lockheed Martin instrument**
  - **single GGI on inertially stabilized platform**
  - **est. airborne precision <10 E/√Hz**

Cessna Grand Caravan

Eurocopter AS-350 B3

all pictures: http://falcon.bhpbilliton.com/falcon/instrumentation.as (obtained in 2007; no longer active)

Airborne Gradiometry - Bell Geospace FTG

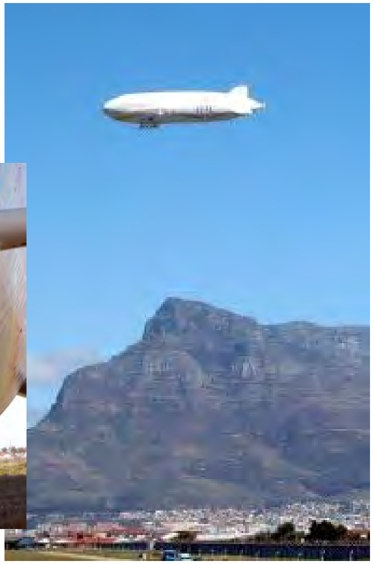
Murphy (2004)

- Full tensor gradiometer (**FTG**)
  - Lockheed Martin instrument
  - This is essentially the **GGSS** (Bell Aerospace)
  - Umbrella configuration of GGIs mounted on **inertially stabilized platform**
  - est. airborne accuracy <10 E/√Hz

ship borne

Cessna Grand Caravan

De Beers Zeppelin-NT



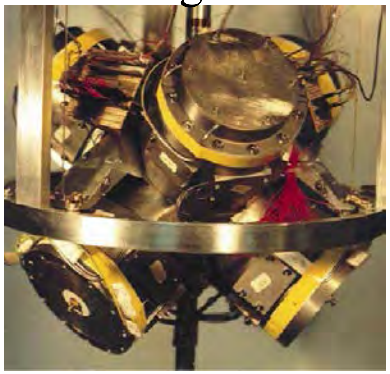
airborne

all pictures: http://www.bellgeo.com/

Other Gradiometers

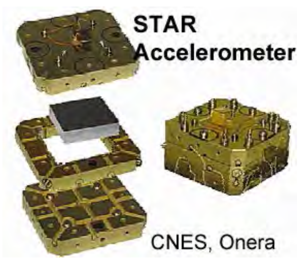
- **Superconducting gradiometer (H.J. Paik, U. Maryland)**

three-axis gradiometer



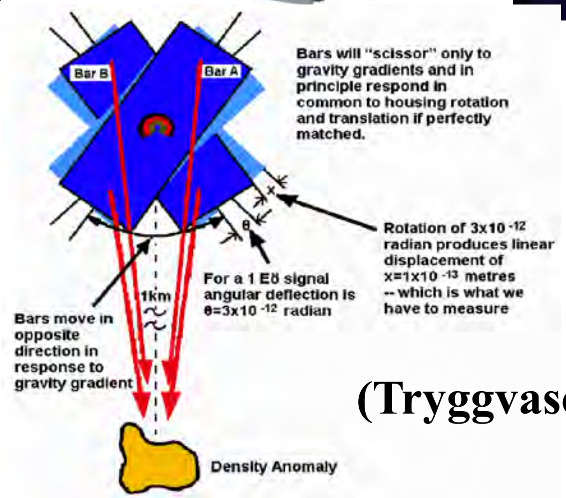
(Paik 2004)

- **GOCE (Gravity Field and Ocean Circulation Explorer)**



true full-tensor gradiometer

- **Gedex Airborne Gravity Gradiometer (AGG)**



(Tryggvason et al. 2004)

- **Cold-atom, interferometer (McGuirk 2001)**

…

Gradiometer System Level Errors

- Gradiometer must detect very small differential gravitation signal within a large-amplitude acceleration environment
  - 1 E accuracy for 0.1 m baseline implies $< 10^{-11}$ m/s^2 accuracy in acceleration

- For gradiometers based on differential accelerometers, scale factor stability and common mode rejection are of highest importance

- Scale factor errors: accelerometers are not perfectly matched

- Alignment errors: sensitive axes of accelerometers are not parallel

- Asymmetry of configuration: measurement point is not center of mass of accelerometer pair

} Special electronic or mechanical devices or procedures are used to eliminate or calibrate these errors
  - induce known dynamics such as rotation or with shakers

- Self-gradients: if system is on a *stabilized* platform, rotation of vehicle about the platform changes the gradient field due to vehicle itself (also changing fuel levels may alter the field significantly)

} Requires calibration of self gradients for different attitudes of vehicle with respect to platform

Errors in Derived Gravitational Gradients

- **Gravitational gradients in body frame assuming true FTG**

$$\boldsymbol{\Gamma}^b = -\mathbf{B}^b + \boldsymbol{\Omega}^b_{ib}\boldsymbol{\Omega}^b_{ib} \qquad \mathbf{B}^b = \frac{1}{2}\left(\frac{\delta \boldsymbol{a}^b}{\delta \boldsymbol{b}^b} + \left(\frac{\delta \boldsymbol{a}^b}{\delta \boldsymbol{b}^b}\right)^{\mathrm{T}}\right)$$

- **Gravitational gradients in *n*-frame**

$$\boldsymbol{\Gamma}^n = -\mathbf{C}^n_b\left(\mathbf{B}^b - \boldsymbol{\Omega}^b_{ib}\boldsymbol{\Omega}^b_{ib}\right)\mathbf{C}^b_n$$

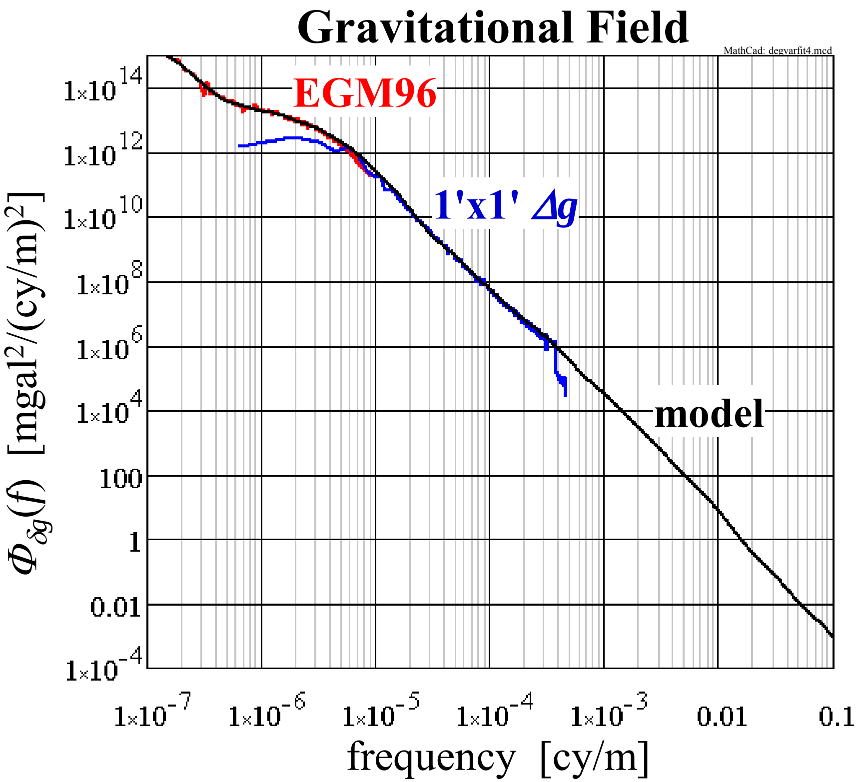
- **Errors represented by linear perturbation:**

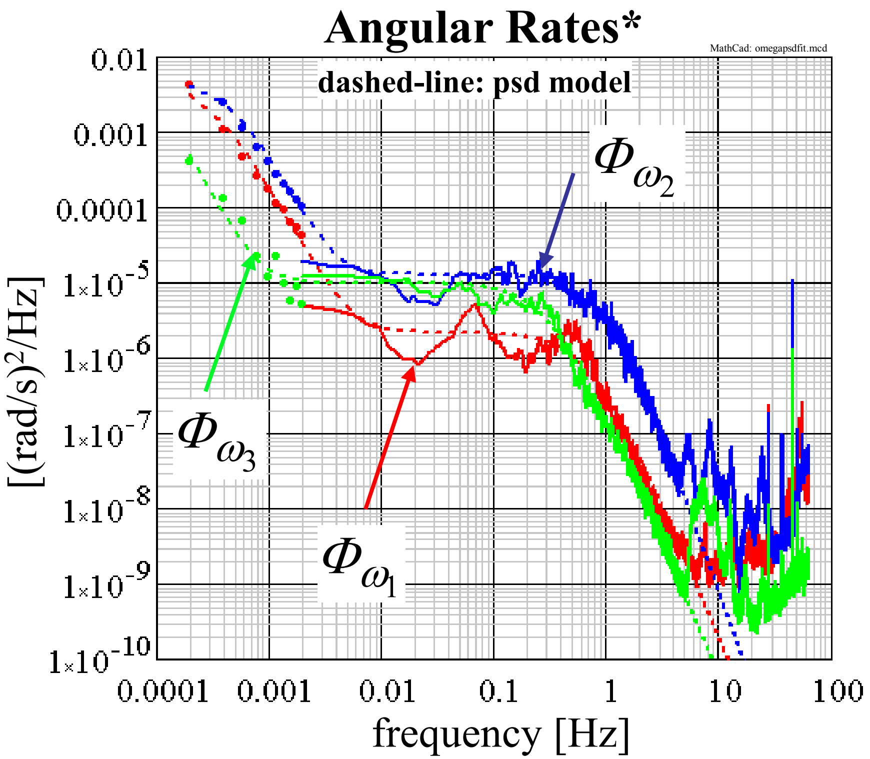
$$\delta\boldsymbol{\Gamma}^n = \boldsymbol{\Gamma}^n\boldsymbol{\Psi} - \boldsymbol{\Psi}\boldsymbol{\Gamma}^n - \mathbf{C}^n_b\left(\delta\mathbf{B}^b - \delta\boldsymbol{\Omega}^b_{ib}\boldsymbol{\Omega}^b_{ib} - \boldsymbol{\Omega}^b_{ib}\delta\boldsymbol{\Omega}^b_{ib}\right)\mathbf{C}^b_n - \underbrace{\sum_j \Xi^n_j \delta x_j}_{\text{negligible}}$$

- $\delta\boldsymbol{\Gamma}^n$: gravitational gradient errors
- $\boldsymbol{\Psi}$: errors in sensor orientation
- $\delta\mathbf{B}^b$: errors in gradiometer measurements
- $\delta\boldsymbol{\Omega}^b_{ib}$: errors in angular rate
- Ξ^n_j: gradients of gravitational gradients
- δx_j: position errors

- **Gradient error PSD is obtained from models of sensor error PSDs and PSDs of gradient field and of angular rates**

PSD Models





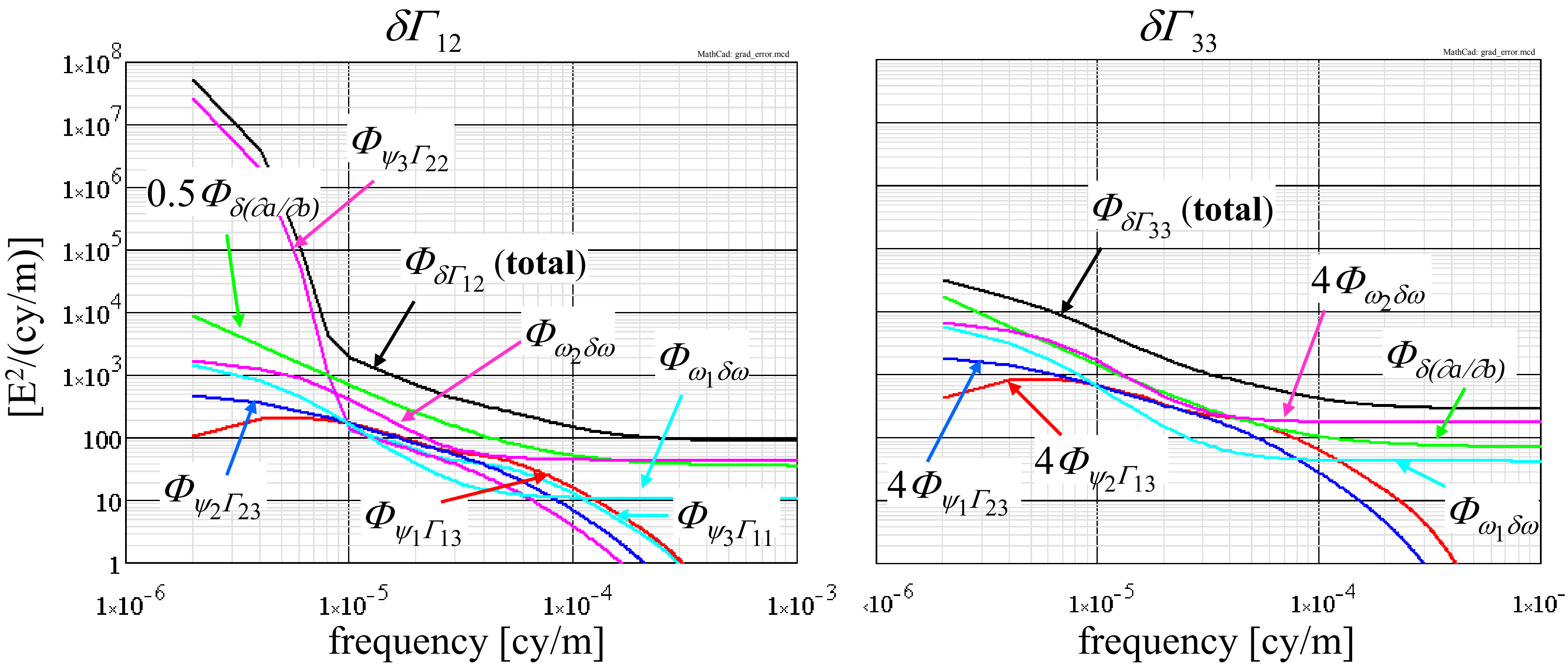
* data from Twin Otter aircraft

- **Gradiometer instrument: white noise**
- **Gyros: rate bias plus white noise**
- **Orientation: initial bias**

} **approximated by simple PSD models**

- **Aircraft parameters are the same as for analysis of inertial (vector) gravimetry**

PSD of $\delta\Gamma_{12}$ and $\delta\Gamma_{33}$ Errors due to "Commensurate" Gyro/Orientation and Gradiometer Errors



System error parameters for commensurate effects

grad noise: 1 E/√Hz	**rate bias: 0.015 deg/hr**
init. orient. s.d.: 0.6 deg	**rate white noise: 0.1 deg/hr/√Hz**

For more details, see (Jekeli, 2003)

Errors in Gravity Gradiometry

Gradiometer White Noise	orientation error	gyro bias	gyro white noise
30 E/$\sqrt{\text{Hz}}$	20 °	0.5 °/hr	3 °/hr/$\sqrt{\text{Hz}}$
10 E/$\sqrt{\text{Hz}}$	6 °	0.15 °/hr	1 °/hr/$\sqrt{\text{Hz}}$
1 E/$\sqrt{\text{Hz}}$	0.6 °	0.015 °/hr	0.1 °/hr/$\sqrt{\text{Hz}}$
0.1 E/$\sqrt{\text{Hz}}$	0.06 °	0.0015 °/hr	0.01 °/hr/$\sqrt{\text{Hz}}$
0.01 E/$\sqrt{\text{Hz}}$	0.006 °	0.00015 °/hr	0.001 °/hr/$\sqrt{\text{Hz}}$

- **Each row corresponds roughly to commensurate sensor errors**
- **Emphasized entries represent typical calibrated error levels for high accuracy airborne systems.**

Gravimetry vs Gradiometry

- Moving-base gravimetry depends on accurate determination of kinematic acceleration, e.g. GPS
  - technology developments in inertial and kinematic acceleration determination are not in synchrony
  - essential limitation in accuracy and resolution
- Moving-base gravity gradiometry depends on accurate angular rate determination
  - accelerometer and gyro technology developments are advancing
  - new technology offers sensors with exquisite sensitivity
  - yields higher resolution; and, with order of magnitude improvement in accuracy, may be useful for change detection
- Gravity gradiometry is better suited to detect high resolution, low-amplitude gravitational signatures
- Practical aspects are similar: Remove terrain effect, perform cross-over adjustment, low-pass filtering