Airborne Gravimetry for Geodesy summer school

Geoid Determination

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The National Geodetic Survey

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Outline

- Brief history of theory of figure of the Earth
- Definition of the geoid
- The geodetic boundary value problems

  - Stokes problem

  - Molodensky problem
- Geoid computations

  -Case study: comparison of different geoid computation methods in the US Rocky Mountains

  - New development in spectral combination

  - xGeoid model of NGS

Outline

The Earth as a hydrostatic equilibrium – ellipsoid of revolution, Newton (1686)

The Earth as a geoid that fits the mean sea surface, Gauss (1843), Stokes (1849), Listing (1873)

The Earth as a quasigeoid, Molodensky et al (1962)

Geoid Definition

Gauss CF - Listing JB

The equipotential surface of the Earth's gravity field which coincides with global mean sea level

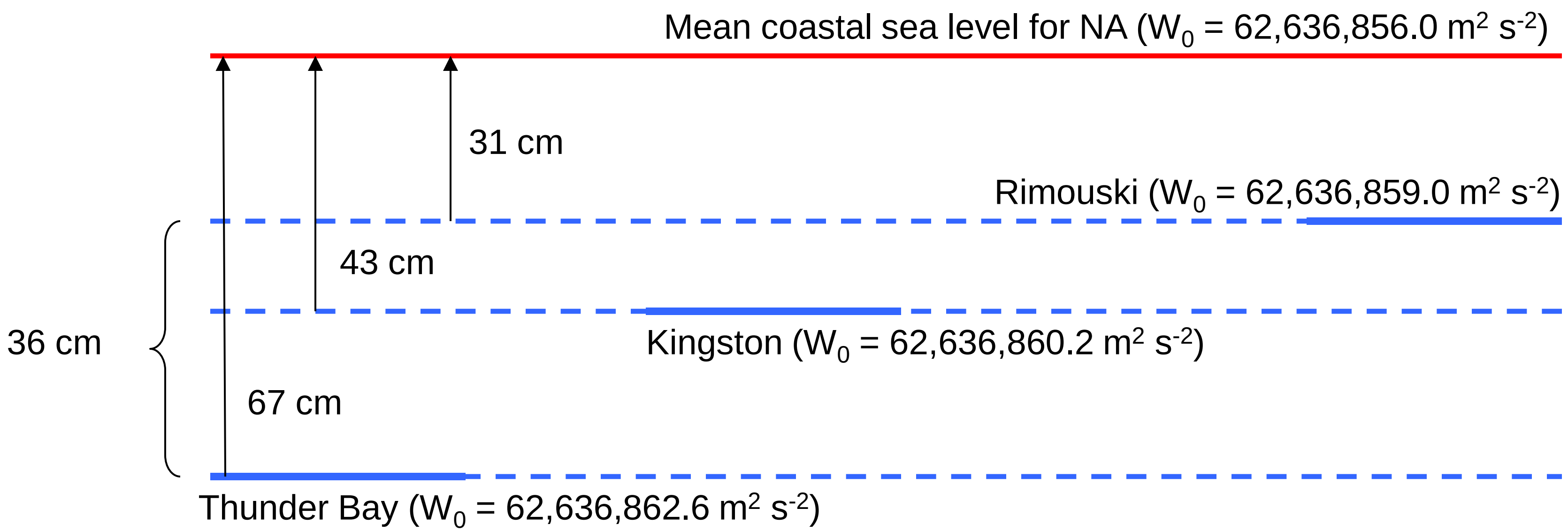
If the sea level change is considered:

The equipotential surface of the Earth's gravity field which coincides with global mean sea level at a specific epoch

Geoid Realization

- **Global geoid**: the equipotential surface ($W = W_0$) that closely approximates global mean sea surface. W_0 has been estimated from altimetric data.
- **Local geoid**: the equipotential surface adopts the geopotential value of the local mean see level which may be different than the global W_0, e.g. $W_0 =$ 62636856.0 $m^2 s^{-2}$ for the next North American Vertical datum in 2022. This surface will serve as the zero-height surface for the North America region.

Different W_0 for N. A. (by M Véronneau)

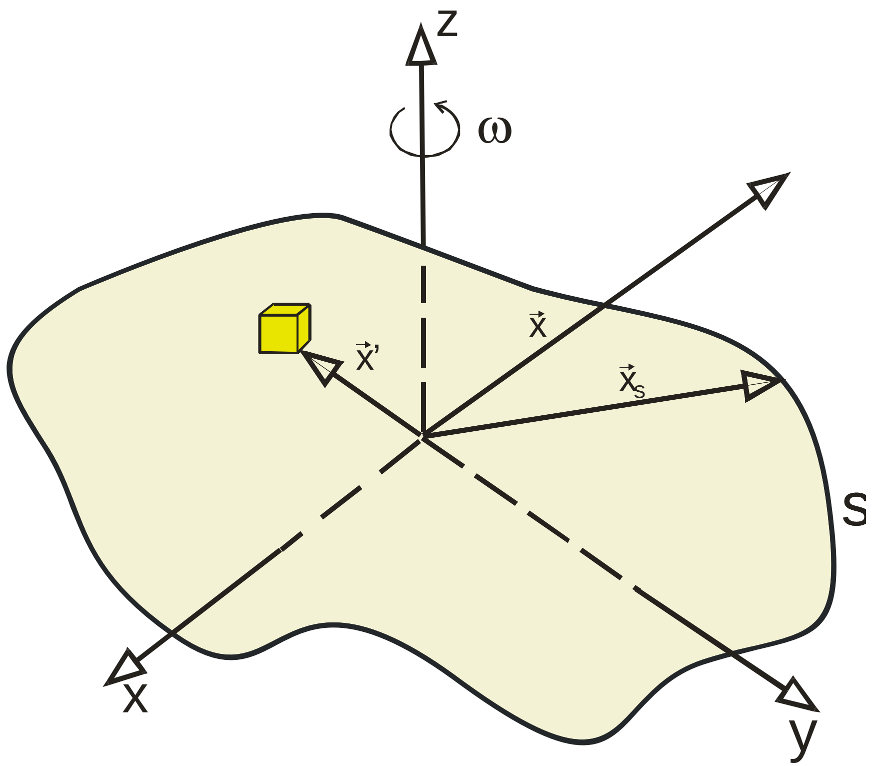


Existing geoid models and reference potential

- ✓ (XGEOID series): 62,636,856.0 $m^2\ s^{-2}$
- ✓EGM2008, USGG2009 and CGG2010: 62,636,855.69 $m^2\ s^{-2}$ (3 cm higher than coastal MSL for NA)
- ✓EGM96, USGG2003 and CGG2005: 62,636,856.88 $m^2\ s^{-2}$ (8 cm lower than coastal MSL for NA)

IERS and IAU conventions: 62,636,856.00 $m^2\ s^{-2}$ (0 cm, same as coastal MSL for NA)

Geodetic boundary value problems



Gravity field: geopotential W

$W = V + Z$

V gravitational potential

Z centrifugal potential

Differential equation

$\Delta V = 0$ in free space; Laplace equation

$\Delta V = -4\pi G\rho$; Poisson equation

Given: $W - W_0$ and gravity vector grad W on the boundary surface S

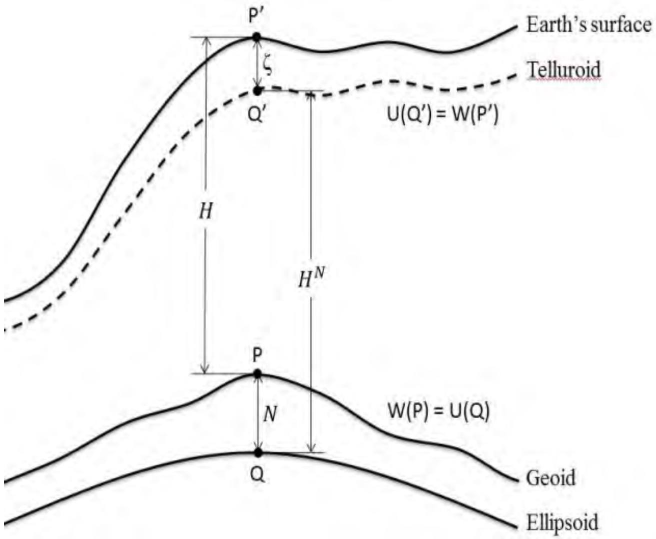
Unknown: W in external space of S and the geometry of S

Basic Concepts and Definitions

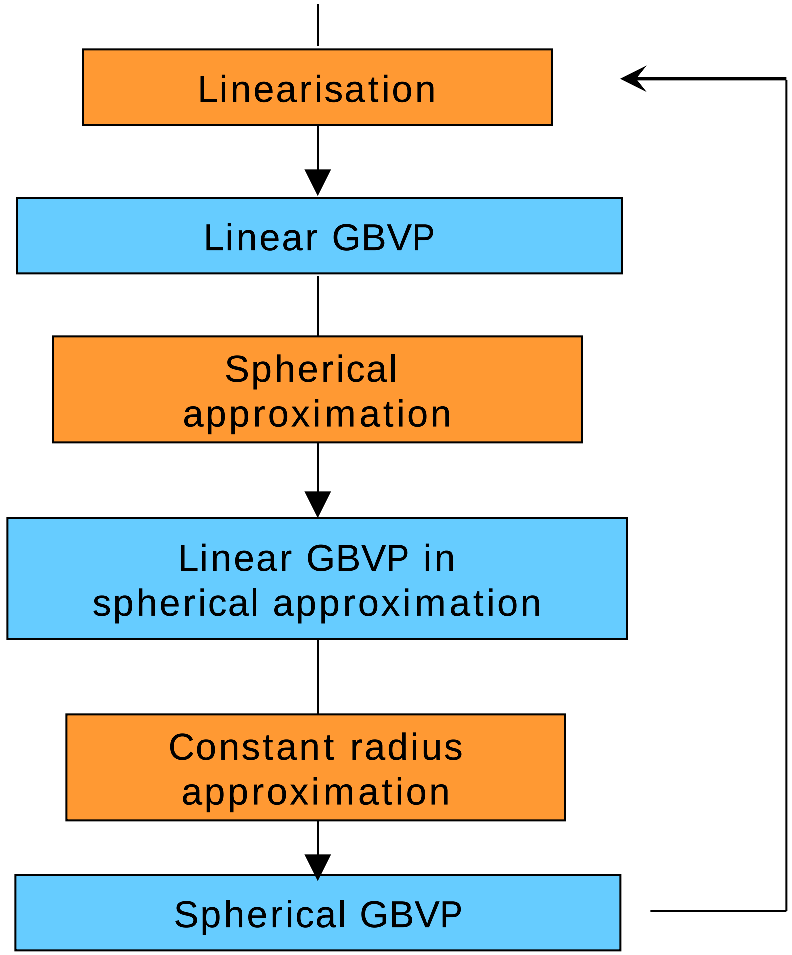
- The normal gravity field U
  - It contains all masses of the Earth
  - It contains the centrifugal potential
  - Mostly used, e.g. GRS80, WGS84
- Disturbing potential T = W - U
  - $\Delta T = 0$ in free space
  - $\Delta T = -4\pi G\rho$ inside masses

Basic Concepts and Definitions

- Normal gravity $$\gamma = -\frac{\partial U}{\partial h}$$
- Gravity anomaly $$\Delta g = g_P - \gamma_Q$$
- Gravity disturbance $$\delta g = g_P - \gamma_P$$
- Geoid height $$N = \frac{T_P}{\gamma_Q}$$
- Height anomaly $$\zeta = \frac{T_{P'}}{\gamma_{Q'}}$$



Flow-chat of solution of GBVP



Approximations: normal potential U

Approximation: $\partial / \partial h \approx \partial / \partial r$

Approximation: $|r| \sim R = \text{const.}$

Analytical solution (integral formula)

Stokes Problem

- Given continuous gravity on the geoid, determine the geometry of the geoid and gravity field above it.

- If the gravity is given on the geoid, the following fundamental geodetic boundary condition can be computed using the normal field:

$$-\Delta g = \frac{\partial T}{\partial h} - \frac{T}{\gamma}\frac{\partial \gamma}{\partial h}$$

- After the spherical approximation, the fundamental geodetic boundary condition becomes

$$-\Delta g = \frac{\partial T}{\partial r} - \frac{2T}{r}$$

Stokes Solution

- The Stokes integral:

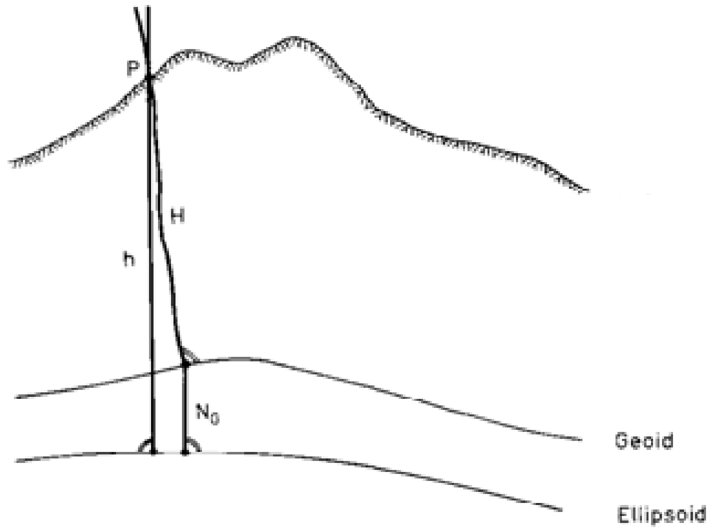
$$N = \frac{R}{4\pi\gamma}\iint_{\sigma} \Delta g S(\psi) d\sigma,$$

$$S(\psi) = \sin^{-1}\frac{\psi}{2} - 6\sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi\ln(\sin\frac{\psi}{2} + \sin^{2}\frac{\psi}{2})$$

* Stokes integral has to satisfy:

1. There is no mass above the geoid (topographic reduction, mean density is often assumed)
2. Data given on the geoid (gravity reduction)
3. Geoid is a sphere (ellipsoidal correction)

Topographic reduction



Density must be known !!!

Removal of topography masses only (Bouguer reduction):

large indirect effect

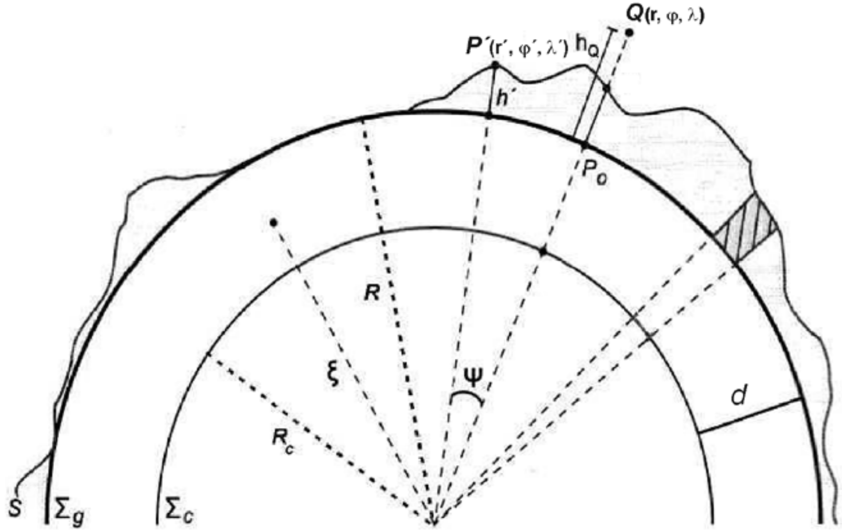
Put topographic masses back

isostatic balance concept

models: Airy-Heiskanen,

Pratt-Hayford, Helmert condensation

- Helmert I:
Condensation of the topographic masses at a surface parallel to the geoid at depth d = 21 km



- Helmert II:
Condensation surface = geoid,
d = 0 km

- Arbitrary depth d of condensation layer

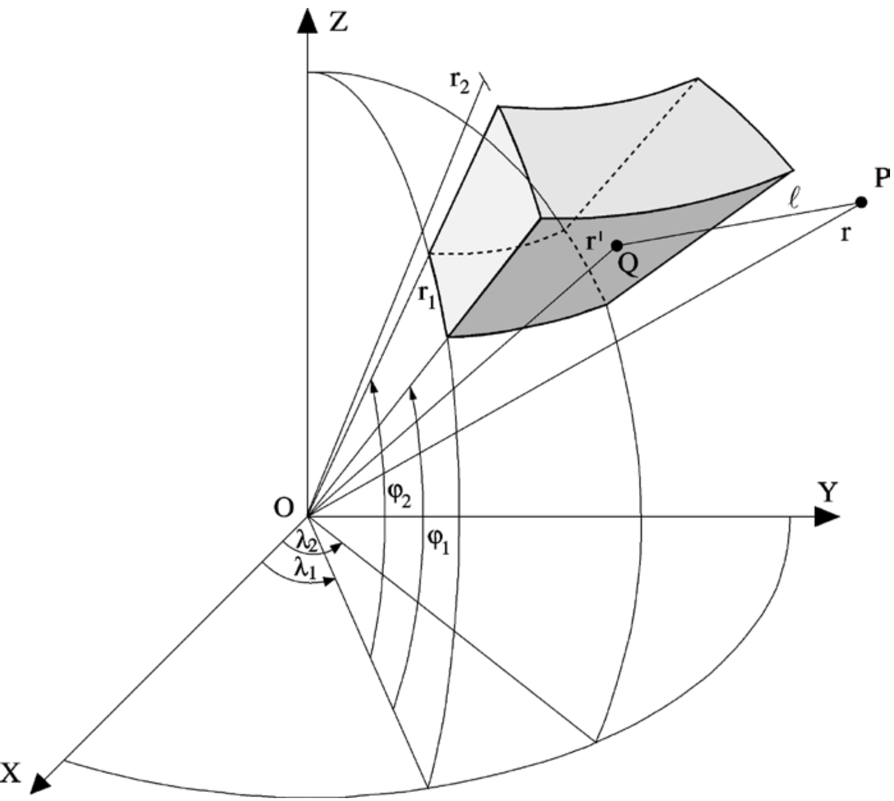
- Principle of mass conservation:

$$\kappa' = \rho \cdot \frac{r'^3 - R^3}{3R_C^2}$$

$$V_C(Q) = G \cdot \iint_\sigma \frac{\kappa'}{\ell_C} \cdot R_C^2 \cdot d\sigma, \quad \ell_C := \sqrt{r^2 + R_C^2 - 2rR_C \cdot \cos\psi}, \quad R_c = R - d$$

Topographic reduction computation (1)

Decomposition of topographic-isostatic masses into mass elements



Digital Elevation Models:

Tesseroids

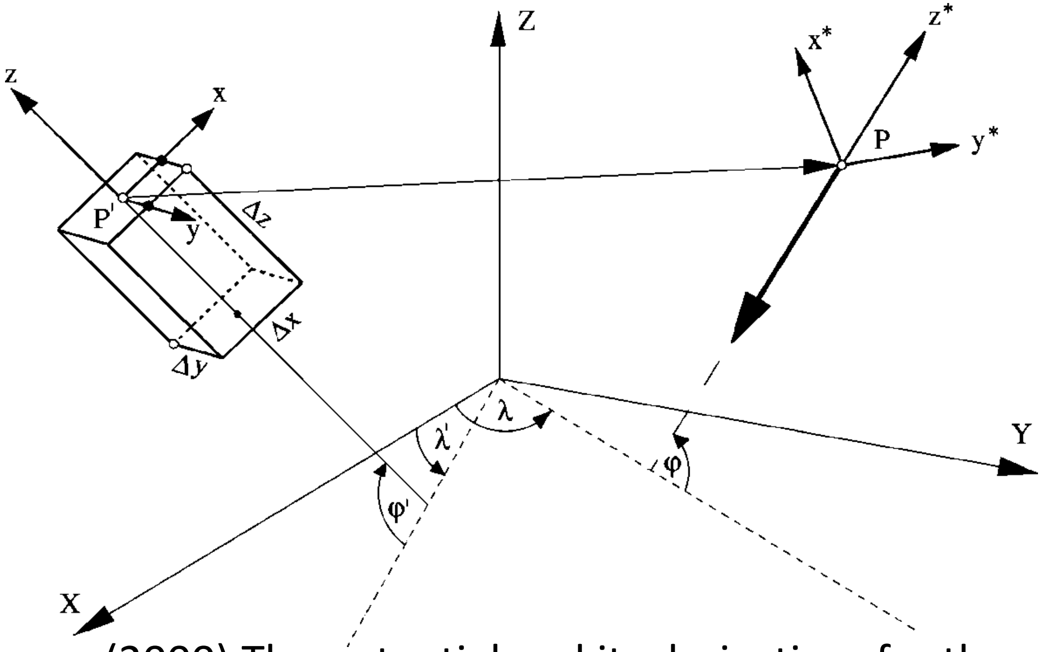
no closed analytical solution
-numerical integration
-Taylor expansion

Heck, B. and Seitz, K. (2007): A comparison of the tesseroid, prism and point-mass approaches for mass reductions in gravity field modelling. JGeod, 81, 121-136.

Topographic reduction computation (1, continued)

Approximation of tesseroids by right-rectangular prisms

Exact analytical solution of 3D integral



Transformation between the edge system of the prism and the local vertical reference frame at the computation point *P*

Nagy D., G. Papp, and J. Benedek: (2000) The potential and its derivatives for the prism. J Geod, 74/7-8, 552 - 560.

Topographic reduction computation (2)

Topographic potential can be reduced into surface integral:

$$V_P(\phi,\lambda) = G\int_\sigma \rho(\phi,\lambda)k(r_S, r_G, r_P, \psi)d\sigma$$

where

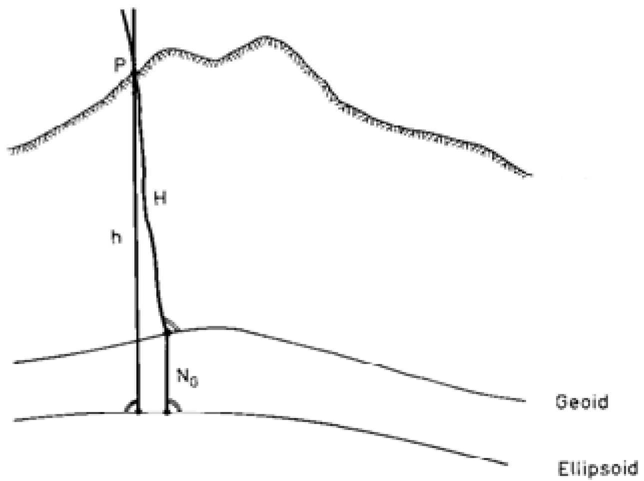
$$k = \left(\frac{3r_P\cos\psi}{2} + \frac{r_S}{2}\right)l_S - \left(\frac{3r_P\cos\psi}{2} + \frac{r_G}{2}\right)l_G + \frac{1}{2}(-1+3\cos^2\psi)r_P^2 \ln\frac{r_S - r_P\cos\psi + l_S}{r_G - r_P\cos\psi + l_G}$$

Expand the kernel into Taylor series for application of 1DFFT

$$k = \left[\frac{3(1+\cos\psi)l}{4} - \frac{R^2}{l}\right]H + \ldots$$

Wang Y.M. (2011) Precise computation of the direct and indirect topographic effects of Helmert's 2nd method of condensation using SRTM30 digital elevation model, *Journal of Geodetic Science,* 2011.

Stokes-Helmert method



H Orthometric height

N Geoidal height

(1) Topographic reduction

Helmert's 2nd method of condensation

(2) Gravity reduction

Harmonic downward continuation, boundary data on the geoid

(3) Stokes integral

Reference model –remove-restore

(4) Indirect effect

Use of the terrain correction

- The Helmert anomaly after the topographic and gravity reductions (harmonic downward continuation) can be approximated by the Faye anomaly: $\Delta g_{Faye} = \Delta g + C$ where *C* is the classical terrain correction
- Then the geoid can be computed approximately using the Stokes integral as

$$N = \frac{R}{4\pi\gamma} \iint_{\sigma} (\Delta g + C) S(\psi) d\sigma + Ind,$$

where *Ind* is the indirect effect. Its first order approximation is $Ind = -\frac{\pi G \rho H^2}{\gamma}$

Free GBVP (Molodensky problem)

Heck catagorized it into two types of free GBVS:

a) Vectorial free GBVP

S completely unknown

Given: $W - W_o$ and gradW on S

Unknown: W in space external of S and position vector of S

b) Scalar free GBVP

S is known by φ, λ (horizontal coordinates)

Given: $W - W_o$ and gradW on S

Unknown: W in space external of S and vertical coordinate (h)

Fixed GBVP

Fixed GBVP

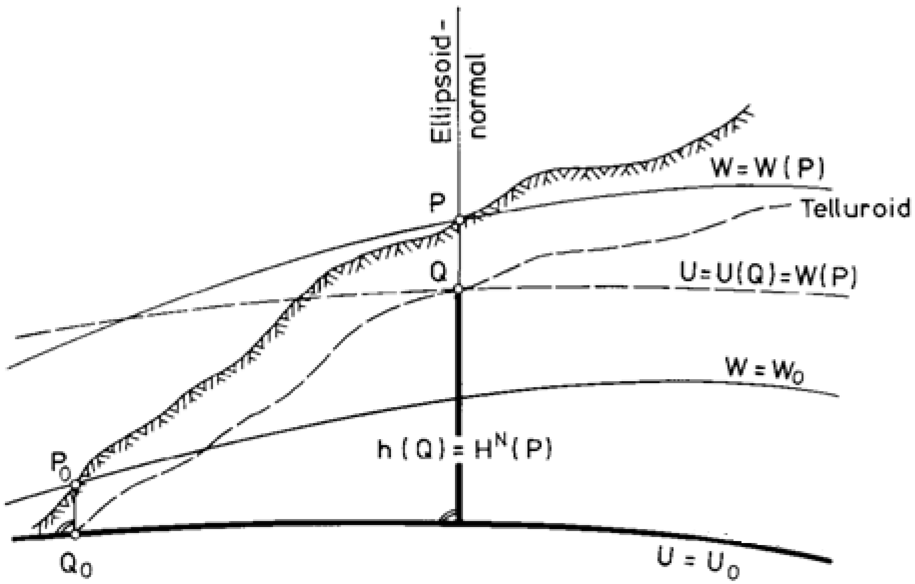
S is known (GPS positioning and remotesensing means)

Given: $W - W_o$ from leveling and the magnitude of gravity $|gradW|$ on S

Unknown: W in space external of S (and in its vicinity)

Key words of Molodensky problem: free, non-linear, oblique

A realization of Molodensky problem (scalar free)



Given on S:

$W(\varphi, \lambda) - W_o$: leveling

Gravity = |gradW|

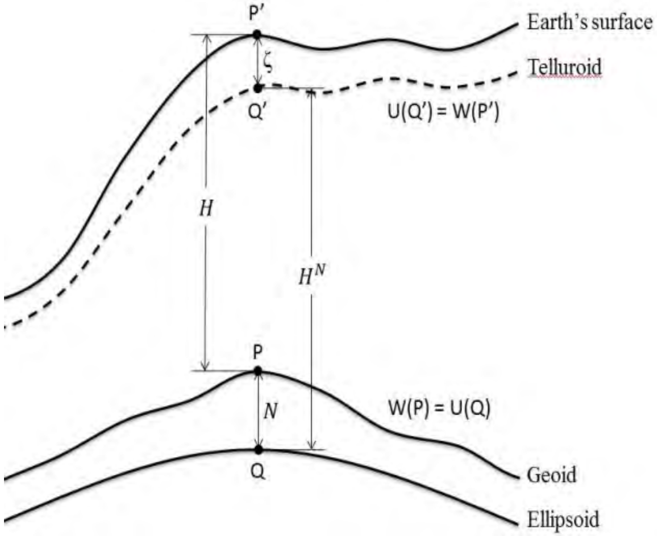
Unknown:

W (X) in space external of S

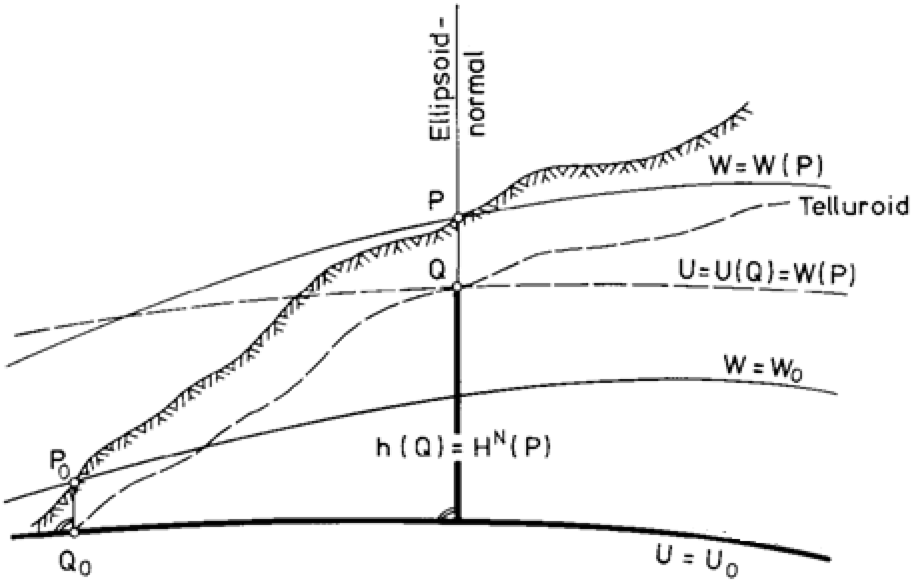
$W = V + Z$

$\Delta V = 0$

Solution of Molodensky problem



- Telluroid as reference surface
- Utilizing a reference gravity filed (e.g. GRS80) and the disturbing potential T(P) = W(P) – U(P) is an small disturbance from the reference (normal) potential U
- Normal height H^N can be computed from the potential number C = W(P) – W_o
- The height anomaly defined as $\zeta = h_P - H^N$



$$\boxed{h = H^N + \zeta}$$

Gravity anomaly:

$$\Delta g := g(P) - \gamma(Q) \qquad (\gamma = |\mathrm{gradU}|)$$

$$= \left(-\frac{\partial \mathrm{T}}{\partial \mathrm{h}} - \frac{2}{r} \cdot T \right)_Q$$

Fundamental GBVP equation

Analytical downward continuation solution

$$T = \frac{R}{4\pi} \cdot \iint_\sigma (\Delta g + g_1 + \ldots) \cdot S(\psi) \cdot d\sigma$$

$$g_1 = \frac{R^2}{2\pi} \iint_\sigma \frac{\Delta g - \Delta g_P}{\ell^3} \cdot d\sigma$$

Bruns formula

$$\zeta = \mathrm{T(Q)} / \gamma(\mathrm{Q})$$

Geoid vs. quasigeoid (1)

Geoid

Poisson differential equation

Lap T = - $4\pi G\rho$

Boundary value problem

downward continuation from the Earth‘s surface onto the geoid

or

Boundary value problem with 2 boundary surfaces

- Earth surface (fixed, known)

- Geoid (free, unknown)

(Grafarend / Martinec)

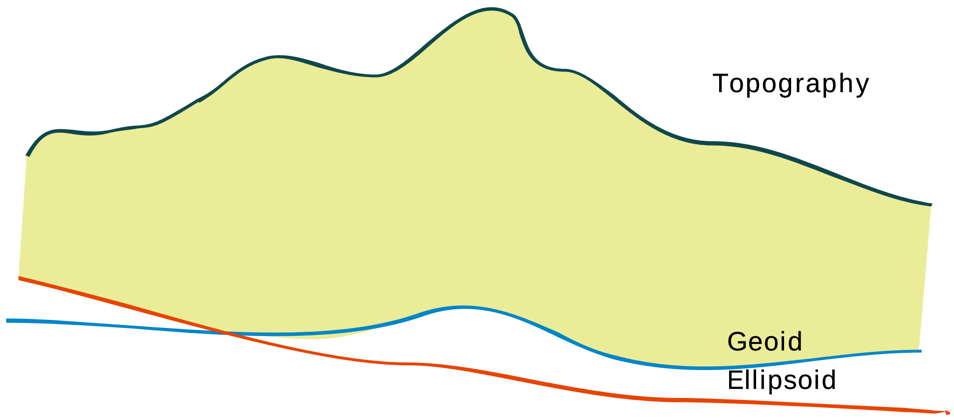
Quasigeoid

Laplace differential equation

Lap T = 0

Boundary value problem

Boundary = Earth surface



Geoid vs. quasigeoid (2)

Geoid	Quasigeoid
Requirement: topographic density and topographic reduction	Independent of density assumptions; density models only for smoothing of the field
Equipotential surface $W = W_o$ and it is a smooth surface	Not an equipotential surface. It is rough in continental regions
Reference surface for orthometric heights	Reference surface for normal heights

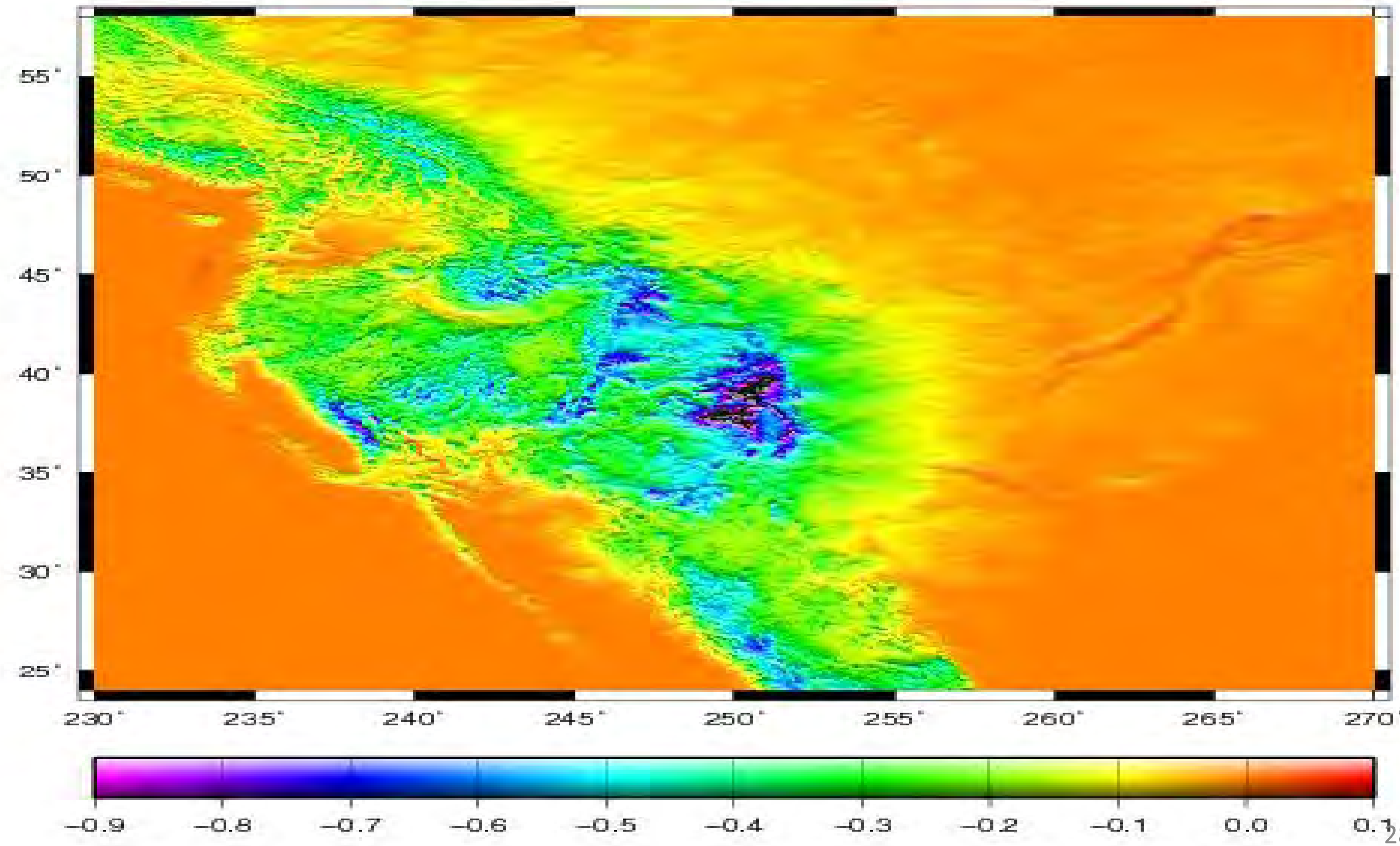
Geoid vs. Quasigeoid

- Geoid-quasigeoid separation

$$\delta = N - \zeta = \frac{\bar{g} - \bar{\gamma}}{\bar{\gamma}} \approx \frac{\Delta g_B H}{\gamma}$$

- More accurate formula see (Flury and Rummel 2009)

Difference between geoid and quasigeoid in USA



Errors in the formulation

Linearisation

non-linear terms in the boundary condition

Spherical approximation

ellipsoidal terms in the boundary condition

topographic terms in the boundary condition

Planar approximation

omission of terms of order (h/R) ~ 10^{-3}

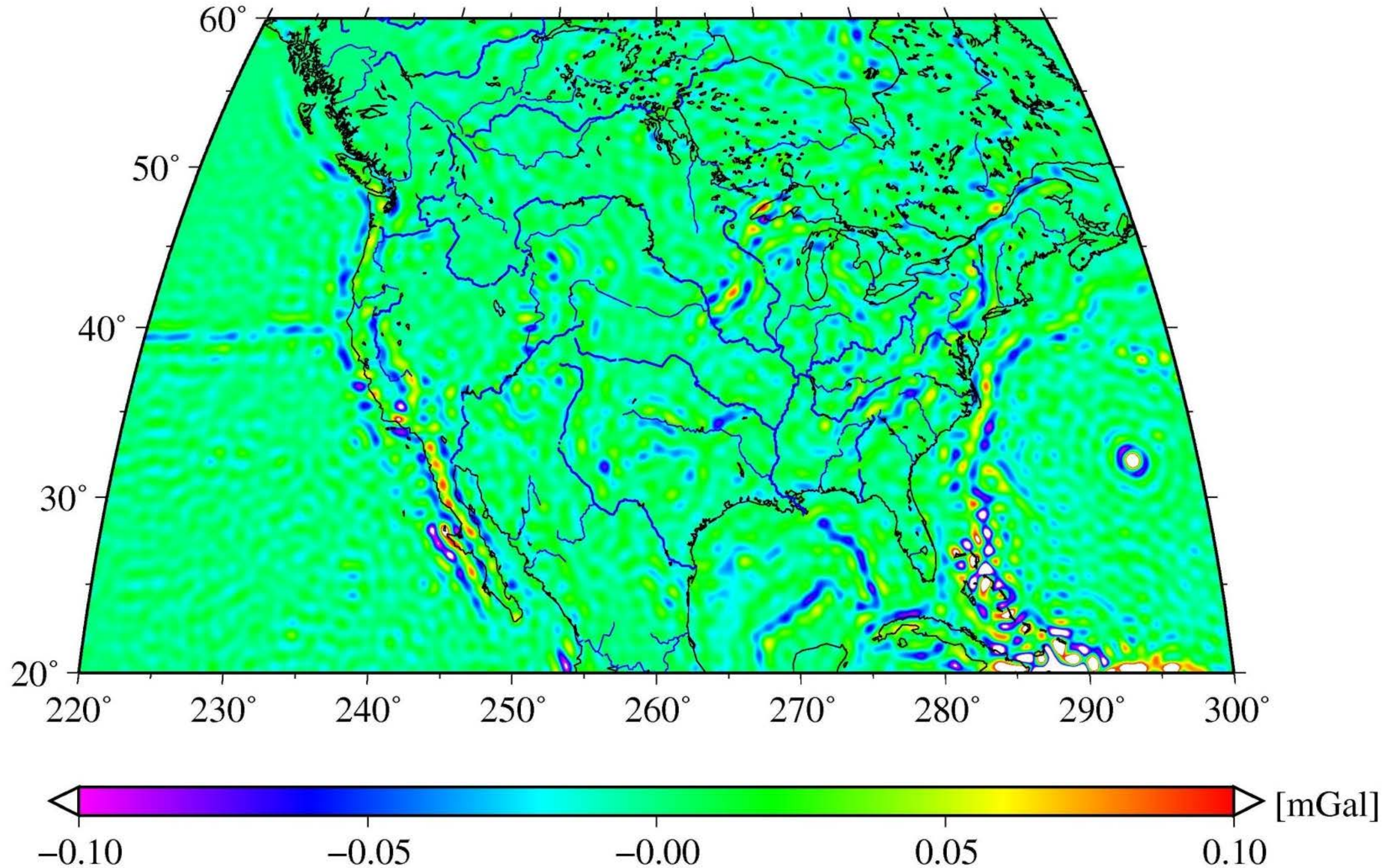
Constant radius approximation

downward continuation effect, Molodensky's series terms

Evaluation of the non-linear boundary condition (North America) (K. Seitz)

- True field ~ EIGEN_GL04C; Nmax = 360
- Topography model: GTOPO30
- Output:
  - non-linear BC
  - non-linear effects in the BC
  - Coordinates of the telluroid points
  - (input for ellipsoidal effects)

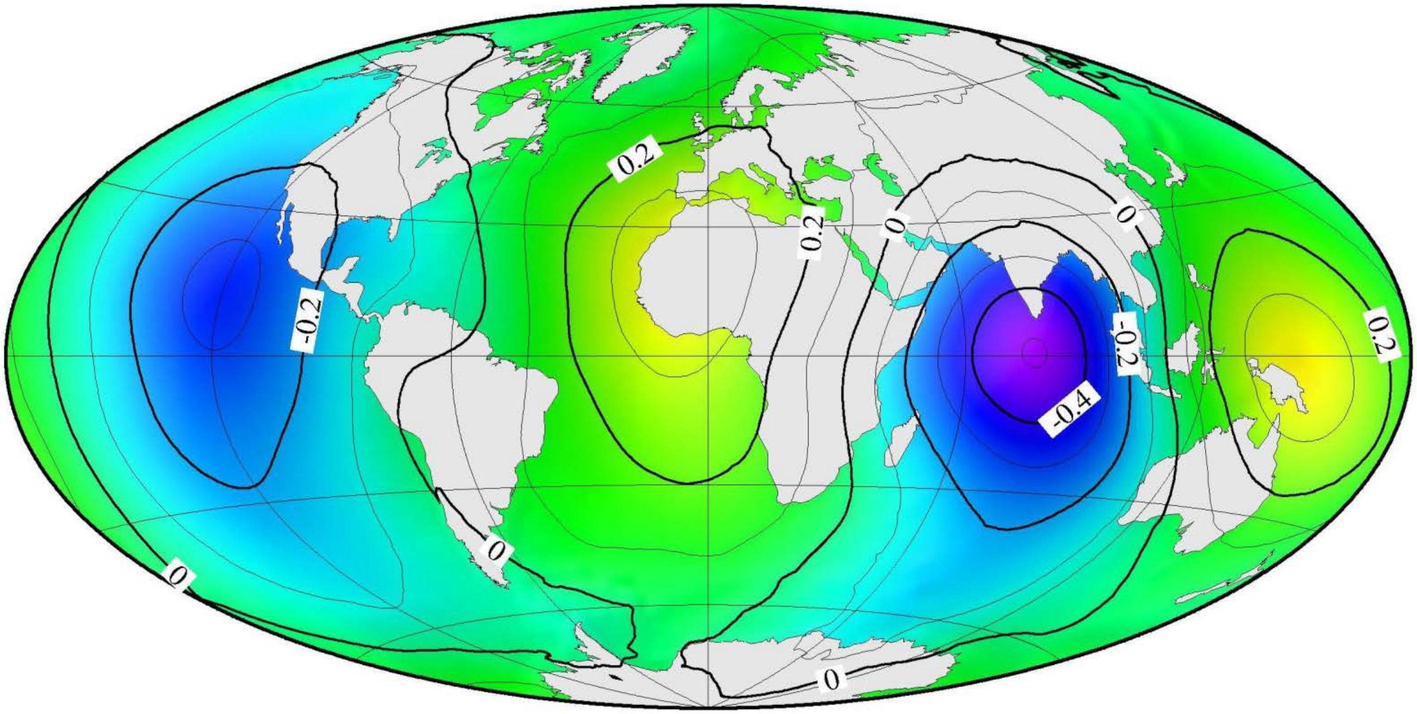
Statistics [mGal]	Min	Max	Mean	L1	L2
Linear BC	-244.885	229.076	-8.246	20.011	25.884
Non-linear BC	-245.197	229.235	-8.246	20.018	25.895
Non-linear eff.	-0.326	0.259	0.000	0.011	0.018

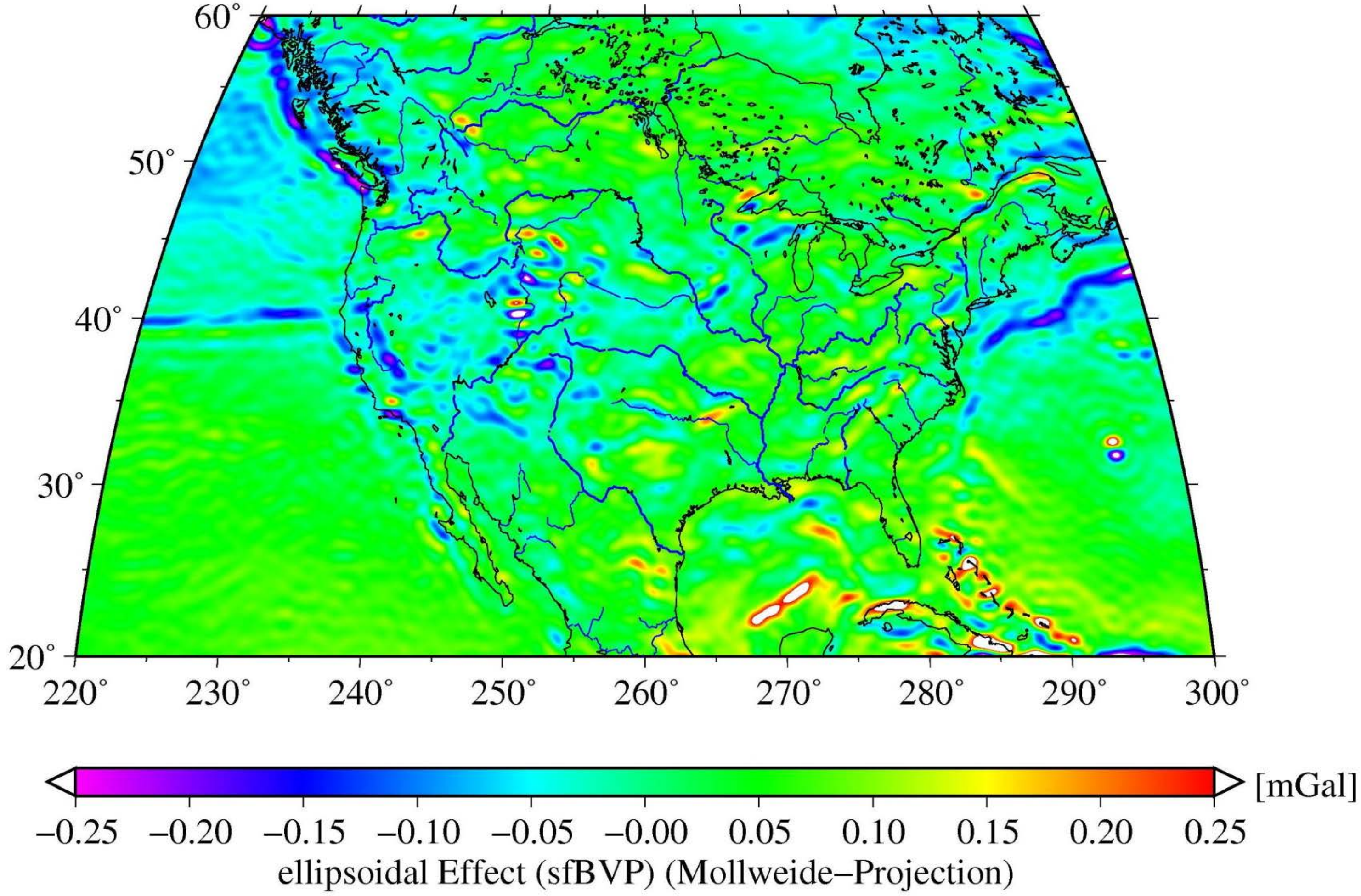


non−linear Effects in the BC (sfBVP) (Mollweide−Projection)

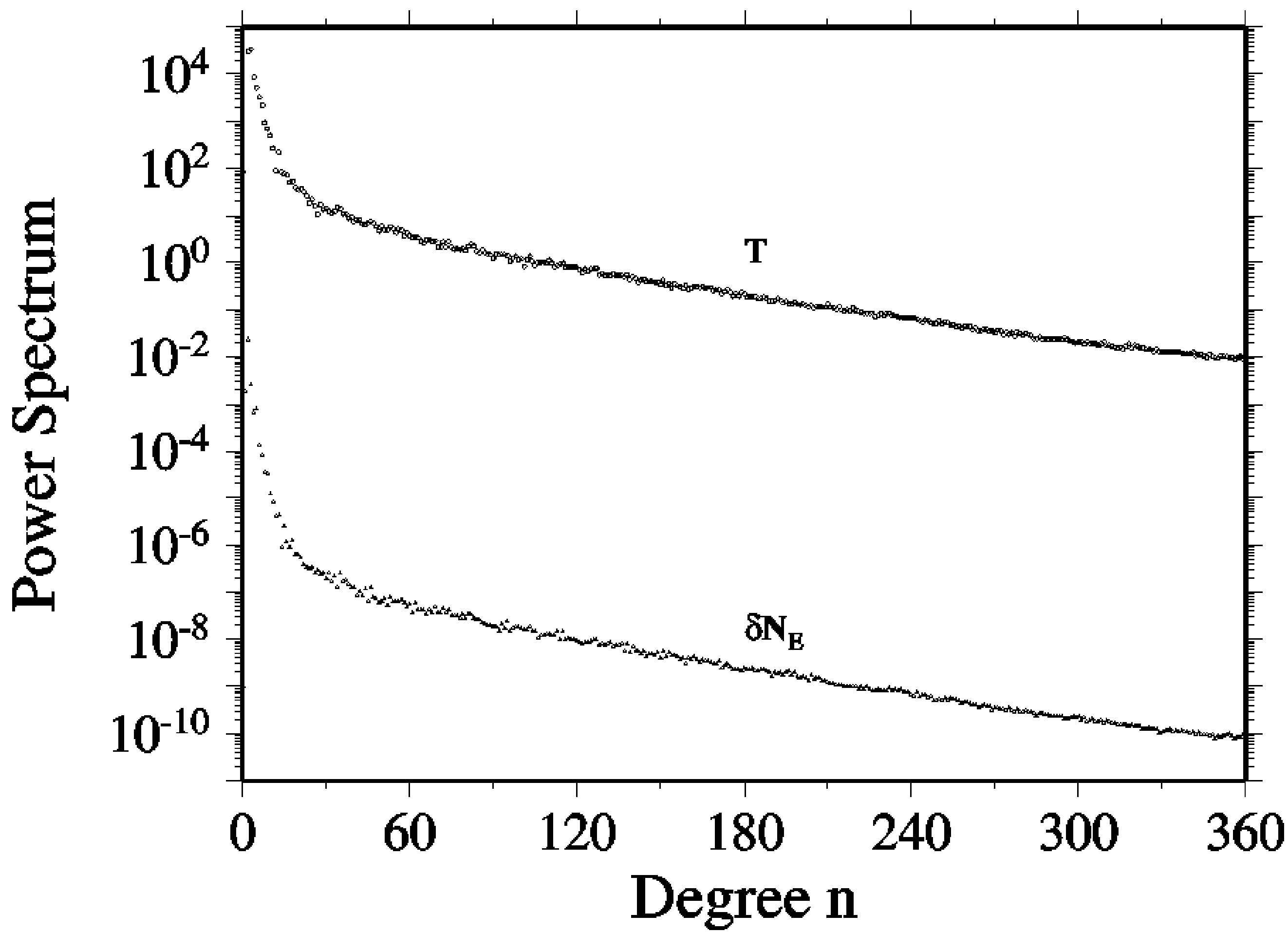
Ellipsoidal correction $\delta N_E = \delta T_E(r_E(\phi,\lambda), \phi, \lambda))/\gamma(\phi)$ in m, $0 \le m \le n \le 360$ (Hammer equal-area projection)

Heck, B. and Seitz, K. (2003): Solutions of the linearized geodetic boundary value problem for an ellipsoidal boundary to order e3. JGeod, 77, 182-192. DOI 10.1007/s00190-002-0309-y.




60°
50°
40°
30°
20°
220°
230°
240°
250°
260°
270°
280°
290°
300°
[mGal]
−0.25
−0.20
−0.15
−0.10
−0.05
−0.00
0.05
0.10
0.15
0.20
0.25
ellipsoidal Effect (sfBVP) (Mollweide−Projection)

Power spectrum of δN_E (in m^2) and T (in m^4s^{-4})



Numerical approximation errors

Evaluation of surface integrals:

- Stokes integral
- Terrain correction
- Molodensky's series terms of higher order
- Poisson integral and derivatives
- ....

Truncation error

Integration over spherical cap, neglection of outer zone

Modified integral kernels

Numerical evaluation by FFT (gridded data)

Finite region - boundary effects, periodic continuation (zero padding)

2D FFT - neglection of sphericity (1D FFT for large regions)

Aliasing, etc.

Comparison of different geoid computation methods in the US Rocky Mountains

Method of harmonic continuation (HDC)

- $dg^{P} = \Delta g^{P} - \Delta g^{P}_{\mathrm{Re}\,f} - \Delta g_{RTM}$
- $N_{Stokes} = \dfrac{R}{4\pi\gamma} \iint_{\sigma} dg^{g} S^{H}(\psi)\, d\sigma$
- $N = N_{\mathrm{Re}\,f} + N_{Stokes} + N_{RTM} + TB$

where

$N_{\mathrm{Re}\,f}$ - Computed from EGM08 with full power

Δg_{RTM}, N_{RTM} - Short wavelength from the topography computed using the program TC

S^{H} - Truncated Stokes kernel (Wong & Gore, l_{max}=120)

TB – topographic bias

Helmert's 2nd method of condensation (H2C)

- $dg^{P} = \Delta g^{P} - A - \Delta g^{P}_{ref\,(Helmertized)}$

- $N_{Stokes} = \dfrac{R}{4\pi\gamma}\iint\limits_{\sigma} dg^{g} S^{H}(\psi)\,d\sigma$

- $N = \zeta^{g}_{ref\,(Helmertized)} + N_{Stokes} + N_{indirect}$

where *A* and $N_{indirect}$ are synthesized from a spherical harmonics expansion to degree 2160; $\zeta^{g}_{ref\,(Helmertized)}$ is computed from the Helmertized EGM2008.

Approximate H2C using the Faye anomaly (H2C_TC)

- $dg^{g} = \Delta g^{P} + TC - \Delta g^{g}_{ref\,(Helmertized)}$

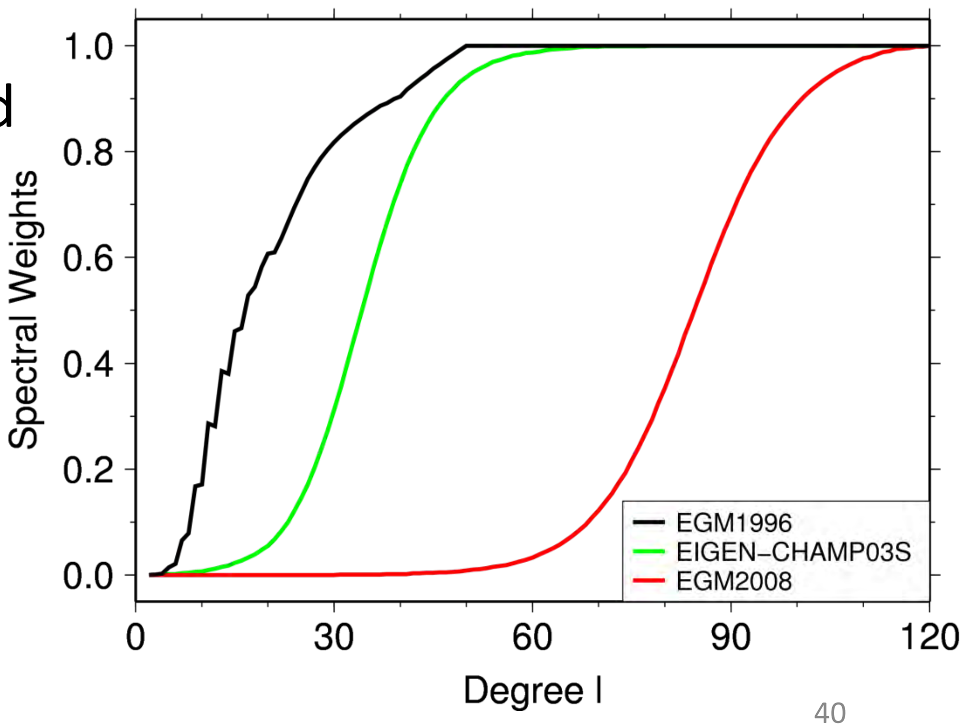
- $N_{Stokes} = \frac{R}{4\pi\gamma} \iint_{\sigma} dg^{g} S^{H}(\psi) d\sigma$

- $N = \zeta^{g}_{ref\,(Helmertized)} + N_{Stokes} + N_{indirect}$

where *TC* is the terrain correction and $N_{indirect}$ is computed using Grushinsky's formula

Geoid from height anomaly (MO, University of Hannover):

- Gravity anomalies defined at the Earth’s surface
- EGM2008 geopotential model (l_{max} = 360)
- RTM reductions (3” terrain model; 15‘ reference topography)
- Gridding by least-squares collocation (1' x 1' grid)
- Spectral combination by 1D FFT
- Height anomalies in a 1' x 1' grid
- Height anomalies converted to geoid undulations using the NGS Bouguer anomaly grid (same gravity data and orth. height used for NAVD88)



Data used

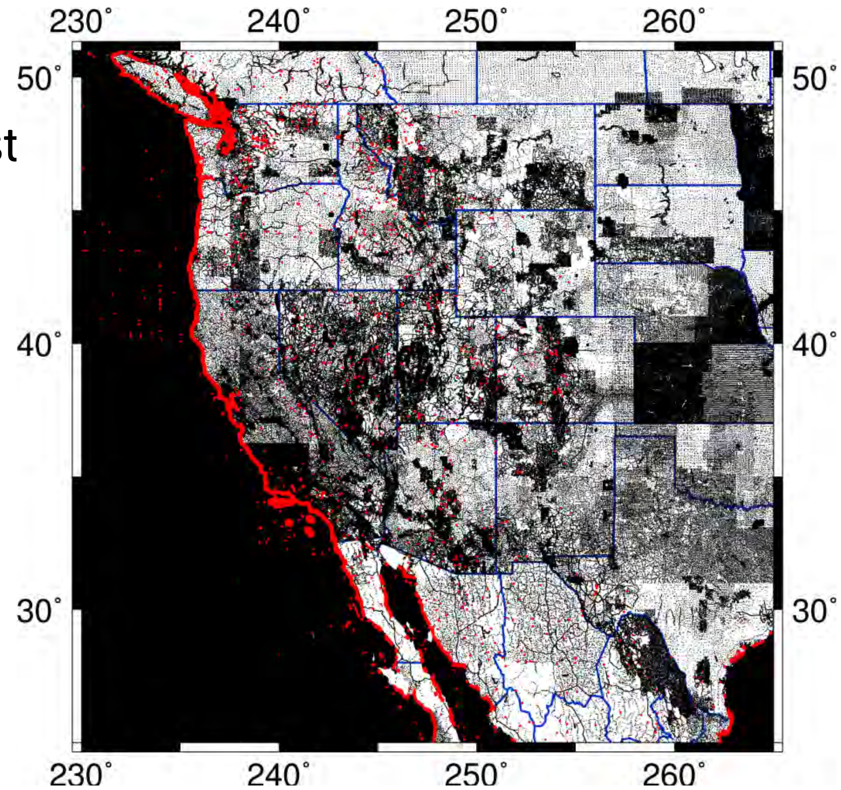
- Over 2 million terrestrial and ship gravity measurements in NGS database + New version of Canadian gravity data
- DNSC08 altimetric gravity anomaly in surrounding oceans.
- 3 arc second Digital Elevation Data (SRTM-DTED1) over the window {10°≤lat≤60° ; 190°≤lon≤308°}
- Global gravity model EGM08 to degree and order 2160

Gravity data editing (NGS)

- The RMS value of the residual free-air anomaly on the Earth's surface is 16.3 mGal for land areas. A few hundred thousands residuals are larger in absolute value than 6 mGal.
- After removing the RTM gravity, the RMS value of the land residuals is reduced to 5.1 mGal.
- All 1341 residuals larger in absolute value than 40 mGal were rejected. Then a K-nearest-neighbor editing rejected 130,800 additional observations.

Gravity data editing (U Hannover)

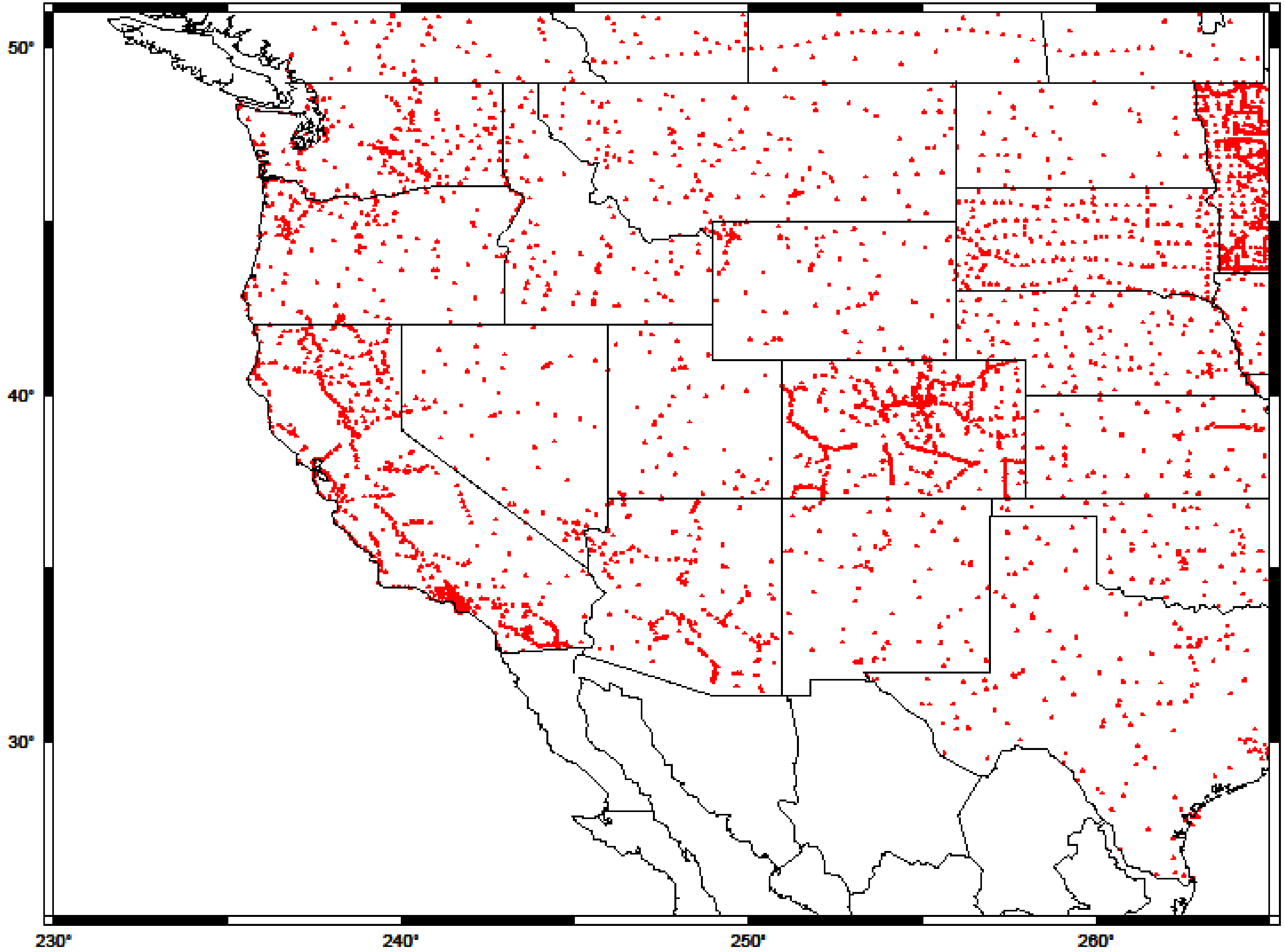
- Check for gross errors
- Editing of the following data:
  - 737 pts. (from DEM comparison)
  - 161 pts. (from 1st check run)
  - 19,774 pts. (altimetry near the coast
  - 723 pts. (from 2nd check run)

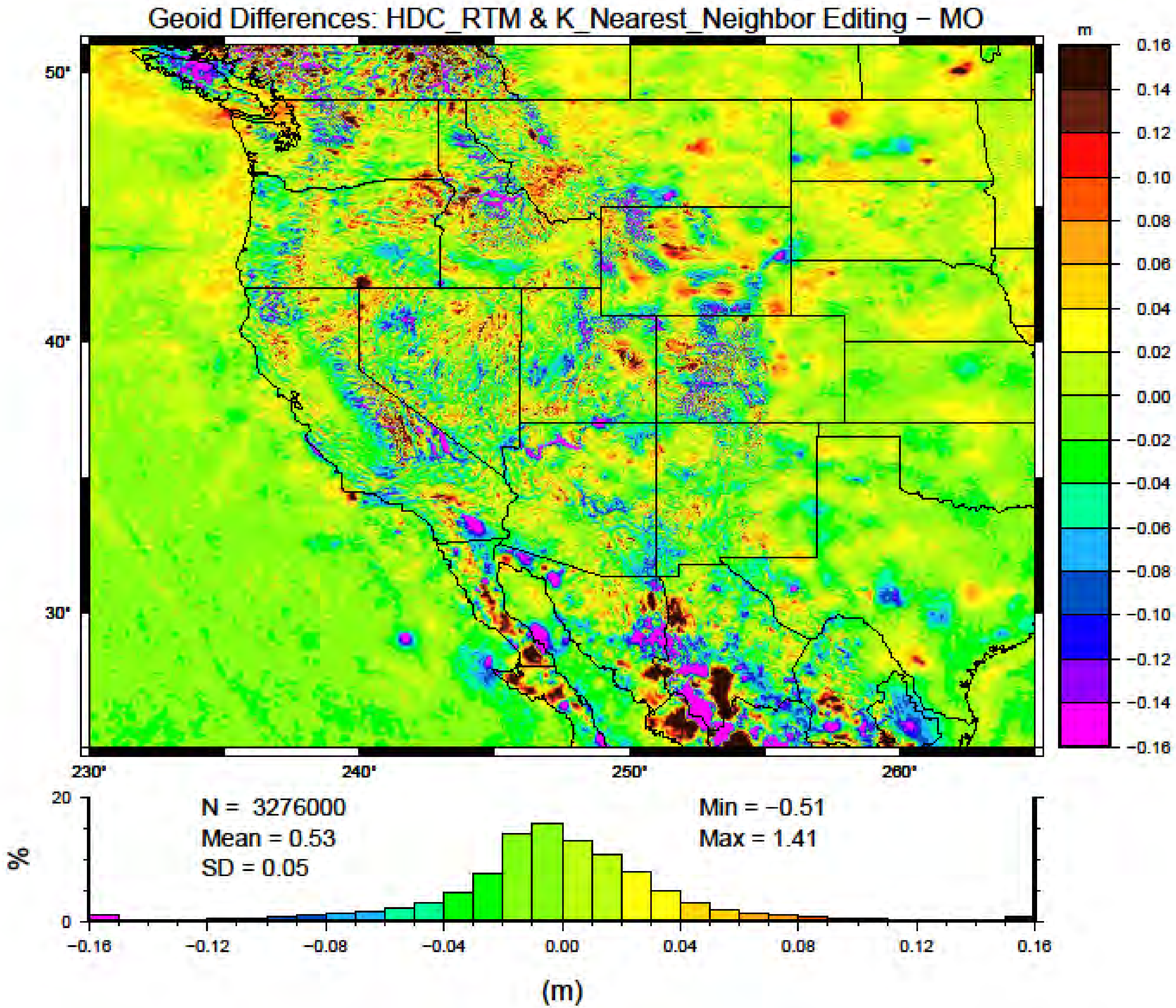


Spherical harmonic expansion of the topographic potential

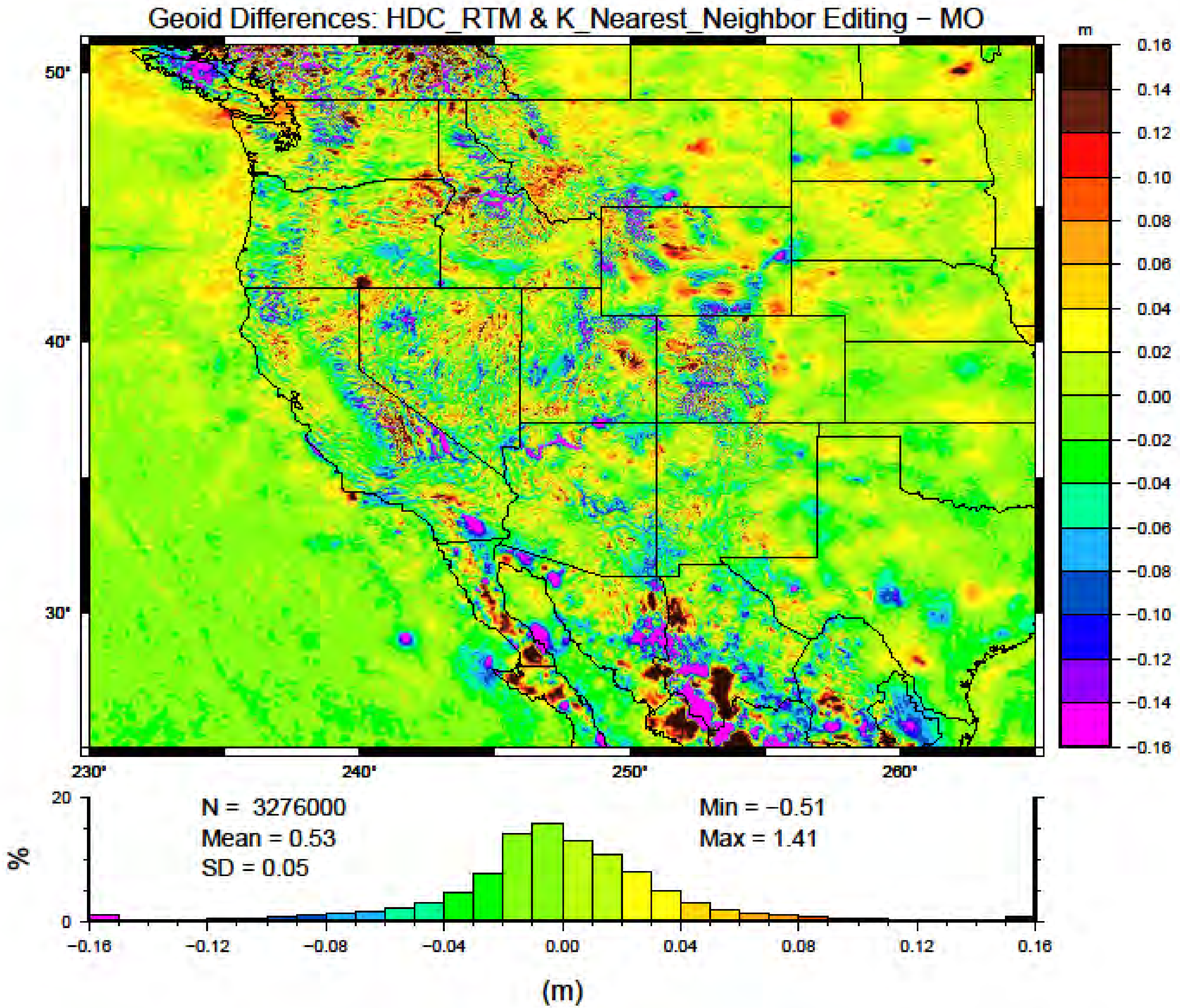
- The spherical approximation is applied
- SRTM DEM is expanded in a S.H. series using quadrature to degree and order 2700
- The zero and 1st degree coefficients are excluded
- The spherical harmonic series is used to compute the direct and indirect effect of Helmert's 2nd method of condensation

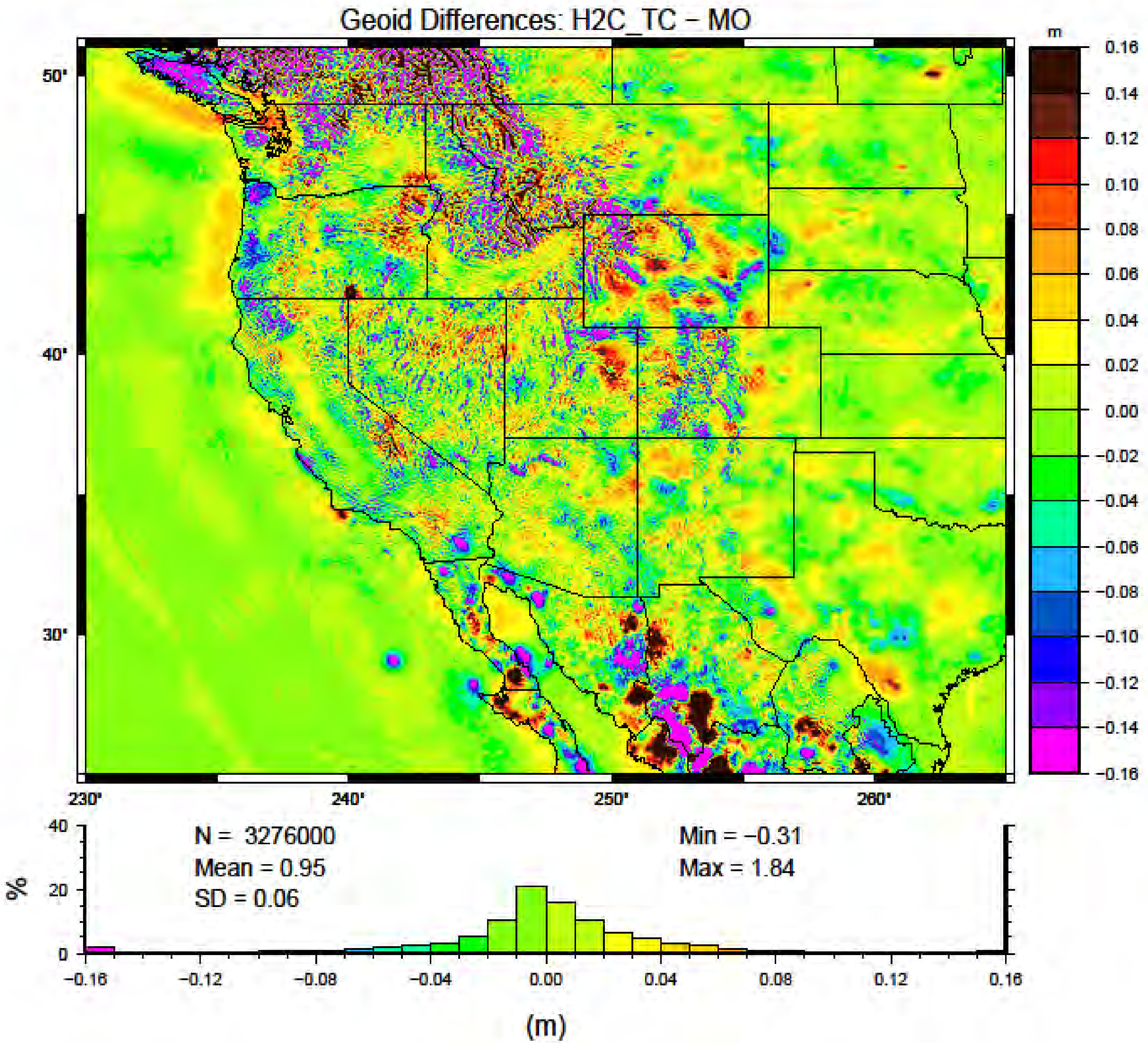
GPS BMs for GEOID09

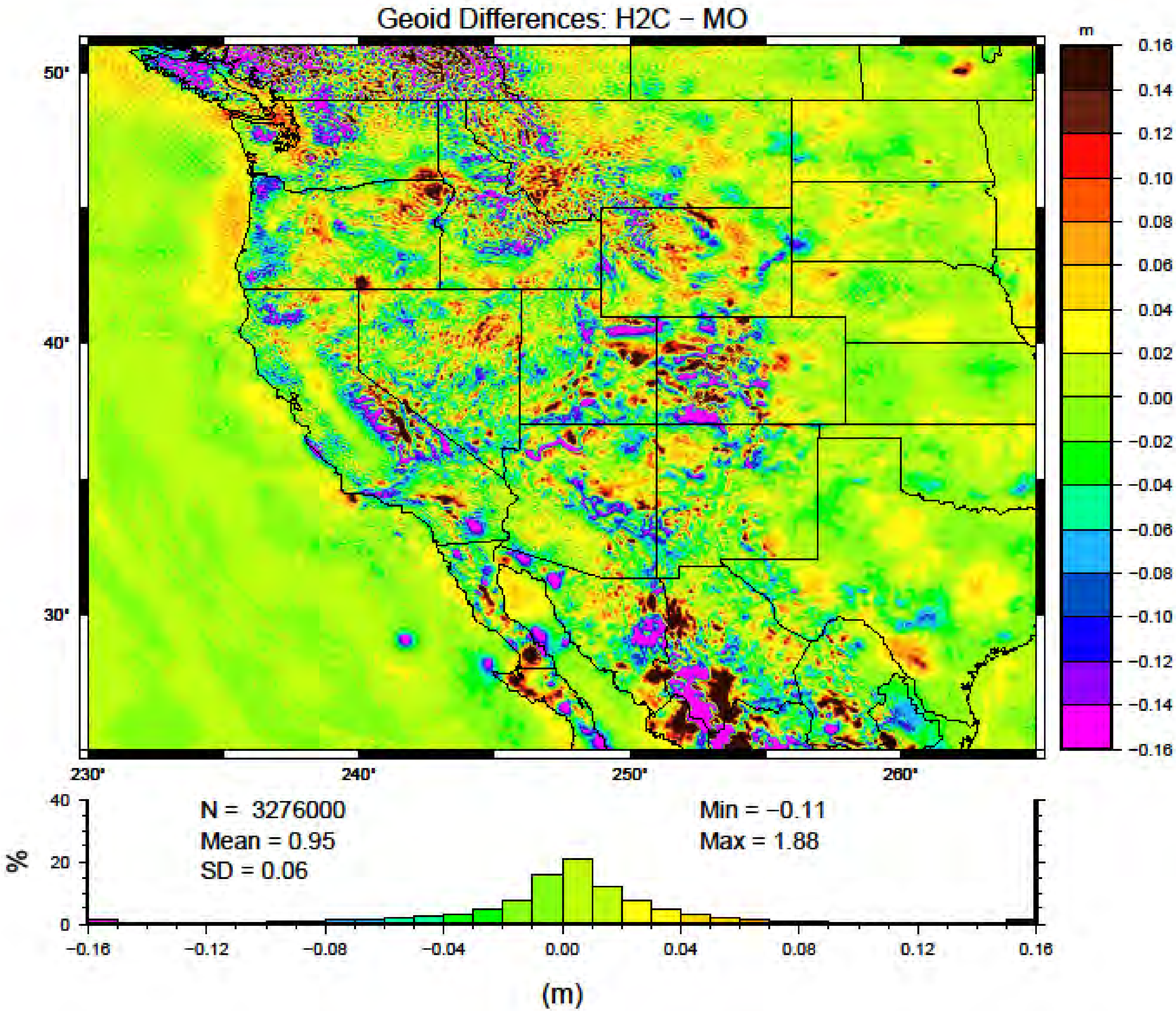


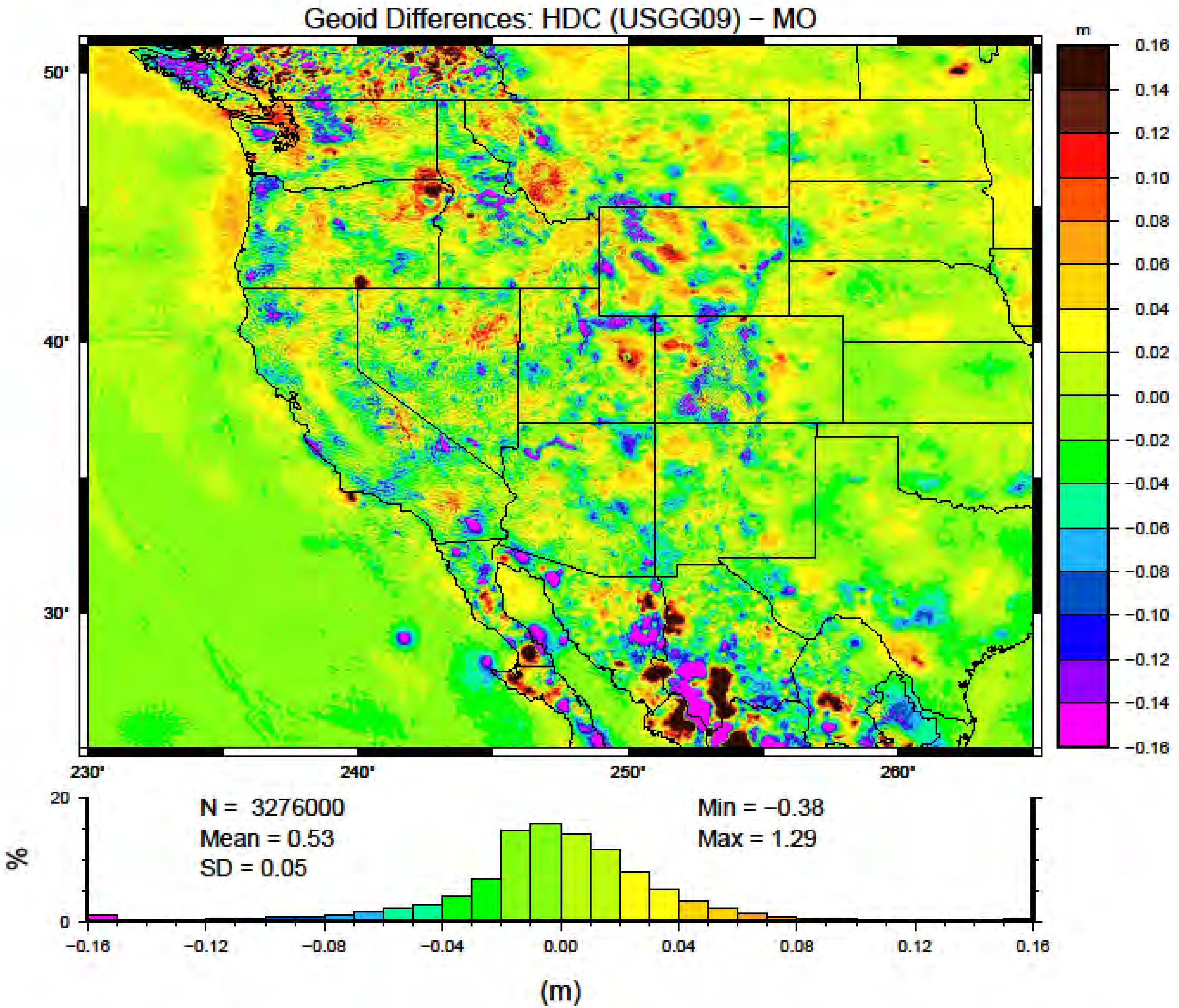


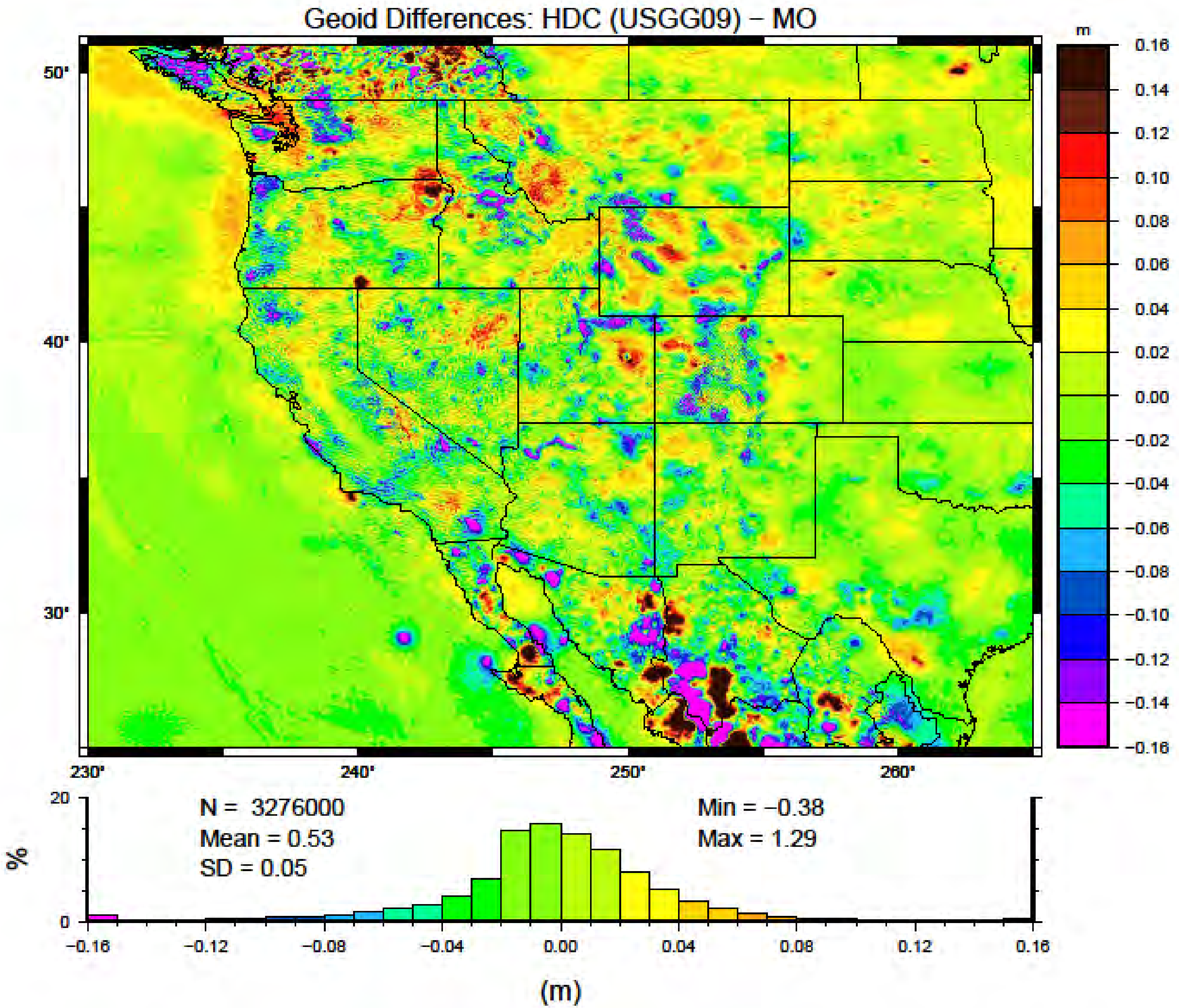
Geoid Differences: HDC_RTM & K_Nearest_Neighbor Editing – MO
m
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
−0.14
−0.16
50°
40°
30°
230°
240°
250°
260°
N = 3276000
Mean = 0.53
SD = 0.05
Min = −0.51
Max = 1.41
%
20
0
−0.16
−0.12
−0.08
−0.04
0.00
0.04
0.08
0.12
0.16
(m)


Geoid Differences: HDC_RTM & K_Nearest_Neighbor Editing − MO
m
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
−0.14
−0.16
50°
40°
30°
230°
240°
250°
260°
N = 3276000
Mean = 0.53
SD = 0.05
Min = −0.51
Max = 1.41
%
20
0
−0.16
−0.12
−0.08
−0.04
0.00
0.04
0.08
0.12
0.16
(m)


Geoid Differences: H2C_TC – MO
m
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
−0.14
−0.16
50°
40°
30°
230°
240°
250°
260°
N = 3276000
Mean = 0.95
SD = 0.06
Min = −0.31
Max = 1.84
%
40
20
0
−0.16
−0.12
−0.08
−0.04
0.00
0.04
0.08
0.12
0.16
(m)


Geoid Differences: H2C − MO
m
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
−0.14
−0.16
50°
40°
30°
230°
240°
250°
260°
N = 3276000
Mean = 0.95
SD = 0.06
Min = −0.11
Max = 1.88
%
40
20
0
−0.16
−0.12
−0.08
−0.04
0.00
0.04
0.08
0.12
0.16
(m)


Geoid Differences: HDC (USGG09) – MO
m
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
−0.14
−0.16
50°
40°
30°
230°
240°
250°
260°
20
%
0
N = 3276000
Mean = 0.53
SD = 0.05
Min = −0.38
Max = 1.29
−0.16
−0.12
−0.08
−0.04
0.00
0.04
0.08
0.12
0.16
(m)



Geoid Differences: HDC (USGG09) – MO
m
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
−0.14
−0.16
50°
40°
30°
230°
240°
250°
260°
20
%
0
N = 3276000
Mean = 0.53
SD = 0.05
Min = −0.38
Max = 1.29
−0.16
−0.12
−0.08
−0.04
0.00
0.04
0.08
0.12
0.16
(m)

Airborne Gravimetry for Geodesy summer school

GPS/leveling comparisons units are in cm

State	No.	H2C	TC	HDC RTM & KNN Editing	HDC(USGG09)	MO(V04)
AZ	242	9.4	8.9	9.6	8.7	9.0
CA	785	13.5	13.2	13.3	13.2	12.2
CO	565	8.8	8.2	8.7	8.3	7.1
ID	97	9.0	8.9	7.5	7.8	7.9
MT	151	10.8	12.4	8.0	9.1	7.8
NV	70	9.5	10.3	8.6	8.8	7.5
NM	107	8.6	9.5	9.3	9.1	8.8
OR	202	8.2	8.1	8.0	8.1	8.2
UT	55	10.0	9.3	8.6	9.0	8.1
WA	259	8.4	9.1	7.0	8.3	7.5
WY	101	9.1	10.3	9.1	8.9	7.5
OK	73	5.7	5.7	5.4	5.7	5.0
KS	100	5.7	5.7	5.5	5.8	6.4
NE	145	4.7	4.9	4.6	4.7	5.0
ND	47	3.2	3.4	3.7	3.3	3.1
SD	242	6.2	6.1	6.2	6.2	5.5
TX	263	8.2	8.4	8.6	8.5	8.6

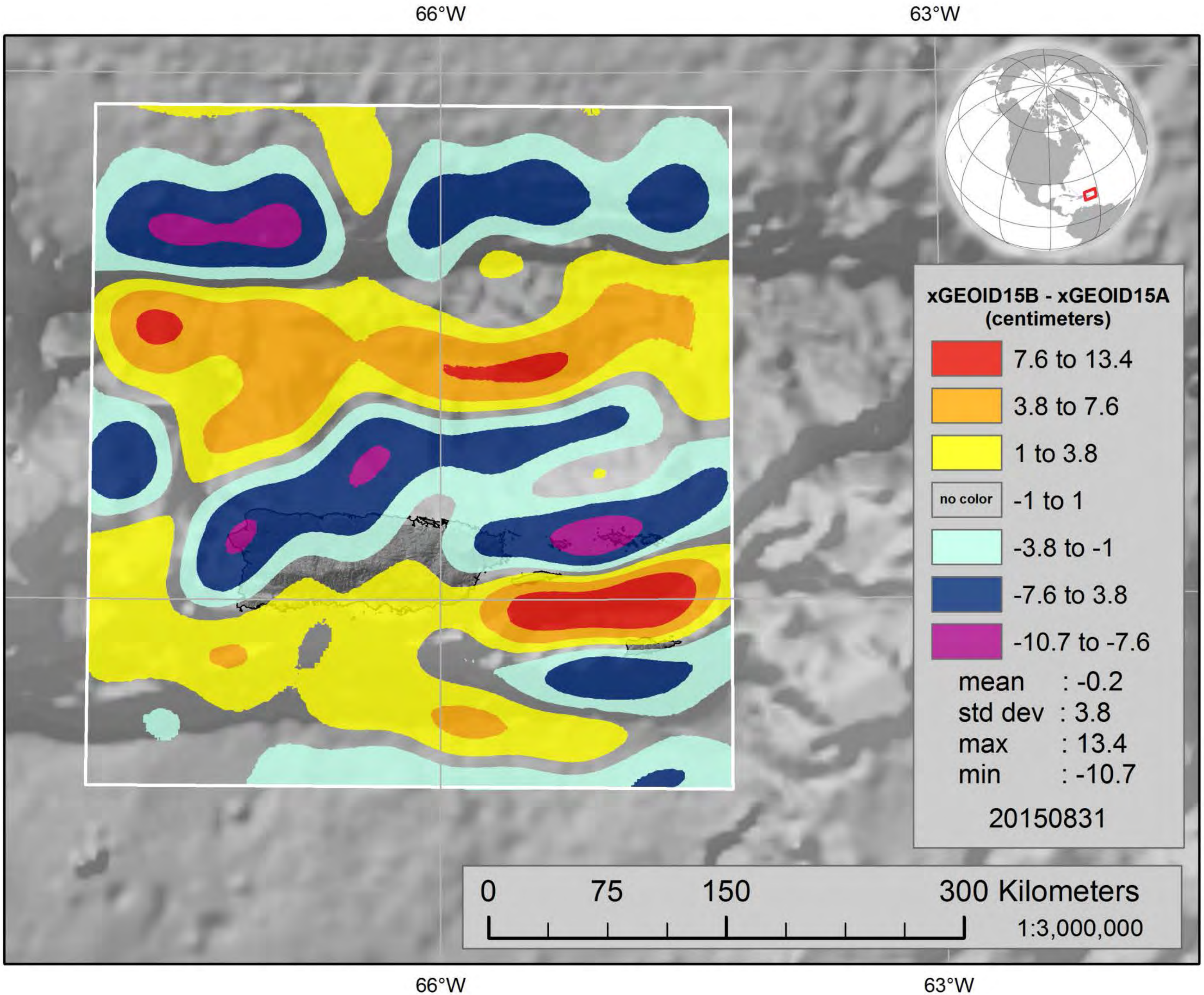
Discussion and conclusions

- Differences between the different geoid solutions are in the range of 5-6 cm in the western mountainous region.
- These differences are mostly due to differences in data weighting, RTM application and use of the reference gravity model.
- GPS/leveling comparisons indicate that the geoid solutions are very comparable and deliver almost the same results.
- The MO performs slightly better than other methods (on the mm level for the whole region).

xGeoid Modeling at NGS

- xGeoid is computed annually to show the improvement bought in by GRAV-D
- Spherical harmonic expansion of GRAV-D data at flight altitude
- Combination of the latest satellite gravity model, the airborne gravity expansion and EGM2008 spectrally – details can be found in the following talk by Dr. Holmes
- Using the airborne gravity enhanced spherical harmonics series as the reference field, compute USGG2009 type of solution – truncation degree based on flight altitude
- Residual terrain model is used in remove-restore fashion.

http://beta.ngs.noaa.gov/GEOID/xGEOID15/xGEOID15_technical_details.shtml


66°W
63°W
21°N
18°N
xGEOID15B - xGEOID15A
(centimeters)
7.6 to 13.4
3.8 to 7.6
1 to 3.8
no color
-1 to 1
-3.8 to -1
-7.6 to 3.8
-10.7 to -7.6
mean : -0.2
std dev : 3.8
max : 13.4
min : -10.7
20150831
0 75 150 300 Kilometers
1:3,000,000

Spectral Combination

Assume *k* sets of data each with global coverage. The spherical harmonic expansion of the *i*th data set is

$$T^i(r,\Omega) = \sum_{n=2}^{N^i_{\max}} \left(\frac{a}{r}\right)^{n+1} \alpha^i_n \phi_n(\Omega)$$

α^i_n coefficients vector

ϕ_n vector of harmonic functions

$N^i_{\max}$ maximum degree and order expansion

Based on the least squares principle, the optimal combination will be a simple weighted mean:

$$T(r,\Omega) = \sum_{n=2}^{N_{\max}} \sum_{i=1}^{k} \left(\frac{a}{r}\right)^{n+1} \omega^i_n \alpha^i_n \phi_n(\Omega)$$

Spectral Combination

where

$$\omega_n^i = \frac{p_n^i}{p_n},\ p_n = \sum_{i=1}^{k} p_n^i \qquad p_n^i = \frac{1}{\sigma_n^i}$$

and σ_n^i is the error degree variance of the coefficients.

An example: spectral combination of terrestrial, airborne gravity data, and a satellite only gravity model.

$$T(r,\Omega) = T^{Sat} + T^{Air} + T^{Terr} = \sum_{n=2}^{N_{\max}} \left(\frac{a}{r}\right)^{n+1} \alpha_n \phi_n(\Omega)$$

where

$$\alpha_n = \omega_n^{Sat}\alpha_n^{Sat} + \omega_n^{Air}\alpha_n^{Air} + \omega_n^{Terr}\alpha_n^{Terr}$$

Equivalence Relationship

On a sphere of radius *a*, the Stokes integral and harmonic series are equivalent (Heiskanen and Moritz 1967 p. 30):

$$T = \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^{n+1} \alpha_n \phi_n = \iint_{\sigma} S'(r,\psi)\Delta g d\sigma$$

where

$$S' = \frac{a}{4\pi} S$$

$$\alpha_n = \frac{GM}{4\pi a(n-1)} \iint_{\sigma} \Delta g(\Omega)\phi_n d\sigma$$

Airborne Gravity Contribution

Therefore, the spherical harmonic series of the airborne gravity is equivalent to the Stokes integral by

$$T^{Air} = \sum_{n=2}^{N_{\max}} (\frac{a}{r})^{n+1} \omega_n^{Air} \alpha_n^{Air} \phi_n = \iint_{\sigma} K^{Air}(r,\psi) \Delta g^{Air} d\sigma$$

where

$$K^{Air} = \frac{a}{4\pi} \sum_{n=2}^{N_{\max}} (\frac{a}{r})^{n+1} \omega_n^{Air} \frac{2n+1}{n-1} (\frac{a+H_{Fly}}{a})^{n+2} P_n(\cos\psi)$$

H_{Fly} average flight altitude

Terrestrial Gravity Contribution

In the same way, the spherical harmonic series of the terrestrial gravity is equivalent to the Stokes integral by

$$T^{Terr} = \sum_{n=2}^{N_{\max}} (\frac{a}{r})^{n+1} \omega_n^{Terr} \alpha_n^{Terr} \phi_n = \iint_{\sigma} K^{Terr}(r,\psi) \Delta g^{Terr} d\sigma$$

where

$$K^{Terr} = \frac{a}{4\pi} \sum_{n=2}^{N_{\max}} (\frac{a}{r})^{n+1} \omega_n^{Terr} \frac{2n+1}{n-1} P_n(\cos\psi)$$

Spectral Combination

Practical consideration:

Airborne and terrestrial gravity are given only locally. How to apply the spectral combination in a most precise way?

Possible solution: Using the relationship between the spherical harmonic series and the global integrals, and using a global gravity model (GGM).

Use of a GGM (1)

The following relationship holds:

$$T_{GGM}^{Air} = \iint_{\sigma} K^{Air}(r,\psi)\Delta g^{GGM}(\Omega')d\sigma'$$

$$= \sum_{n=2}^{N_{GGM}} \left(\frac{a}{r}\right)^{n+1} \left(\frac{a+H_{Fly}}{a}\right)^{n+2} \alpha_n^{GGM} \omega_n^{Air} \phi_n(\Omega)$$

$$T_{GGM}^{Terr} = \iint_{\sigma} K^{Terr}(r,\psi)\Delta g^{GGM}(\Omega')d\sigma'$$

$$= \sum_{n=2}^{N_{GGM}} \left(\frac{a}{r}\right)^{n+1} \alpha_n^{GGM} \omega_n^{Terr} \phi_n(\Omega)$$

where Δg^{GGM} α_n^{GGM} N_{GGM} are gravity anomaly, coefficients vector and maximum degree of expansion of the GGM.

Use of a GGM (2)

Using a GGM, the contribution of airborne and terrestrial gravity can be written as

$$\delta T^{Air} = \iint_{\sigma} K^{Air}(\psi)(\Delta g^{Air} - \Delta g^{GGM})d\sigma$$

$$\approx \iint_{\sigma_0} K^{Air}(\psi)(\Delta g^{Air} - \Delta g^{GGM})d\sigma$$

$$\delta T^{Terr} \approx \iint_{\sigma_0} K^{Terr}(\psi)(\Delta g^{Terr} - \Delta g^{GGM})d\sigma$$

If the integration area is larger than one degree radius and the GGM is higher than 360, the truncation error (omitting the contribution of the rest of the area) is on the mm level.

Put Together...

The geoid can be computed as

$$N = \gamma^{-1}[T^{Sat} + (T_{GGM}^{Air} + \delta T^{Air}) + (T_{GGM}^{Terr} + \delta T^{Terr})] + \delta$$

Or

$$N = \gamma^{-1}(T^{Coef} + \delta T^{Air} + \delta T^{Terr})$$

where

$$T^{Coef} = T^{Sat} + T_{GGM}^{Air} + T_{GGM}^{Terr}$$

and δ is the geoid-quasigeoid separation.

Remove-Restore Scheme

If we set

$\omega_n^{Sat}=0 \quad \omega_n^{Air}=0 \quad \omega_n^{Terr}=1$ for all *n*, (no use of satellite gravity model and airborne gravity)

Then we have

$$N=\gamma^{-1}(T^{Coef}+\delta T^{Terr})+\delta$$

where

$$T^{Coef}=T_{\mathrm{Re}f}^{Terr}=\sum_{n=2}^{N_{\mathrm{Re}f}}(\frac{a}{r})^{n+1}\alpha_n^{\mathrm{Re}f}\phi_n$$

This is nothing but the widely used remove-restore scheme.

Notice the error in gravity data are not reduced.

Wong-Gore Kernel Truncation

If we set

$$\omega_n^{Sat} = 1 \quad \omega_n^{Air} = 0 \quad \omega_n^{Terr} = 0 \quad \text{for } n \le N'$$

$$\omega_n^{Sat} = 0 \quad \omega_n^{Sat} = 0 \quad \omega_n^{Terr} = 1 \quad \text{for } n > N'$$

where *N'* is the degree of choice, then the spectral combination becomes the Wong-Gore kernel truncation:

$$N = \gamma^{-1}(T^{Coef} + \delta T^{Terr}) + \delta$$

where

$$T^{Coef} = \sum_{n=2}^{N'} \left(\frac{a}{r}\right)^{n+1} \alpha_n^{Sat} \phi_n + \sum_{n=N'+1}^{N_{Max}} \left(\frac{a}{r}\right)^{n+1} \alpha_n^{GGM} \phi_n$$

$$\delta T^{Terr} \approx \iint_{\sigma_0} K^{Terr}(\psi)(\Delta g^{Terr} - \Delta g^{GGM})d\sigma$$

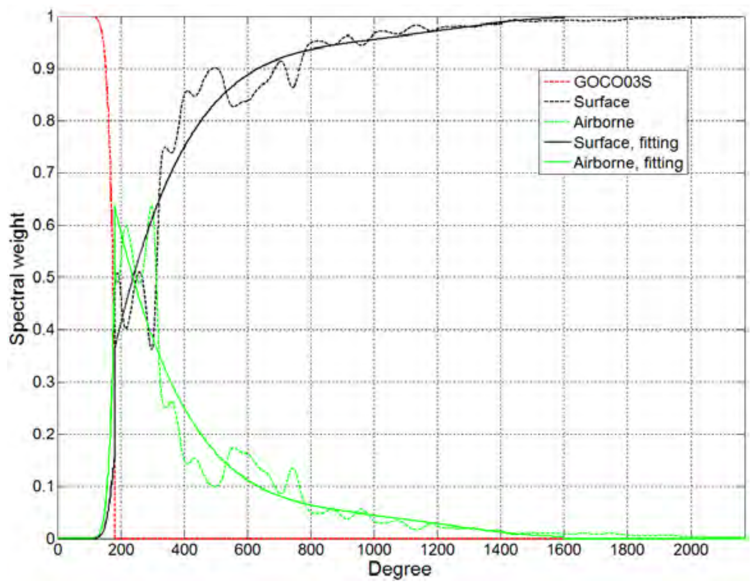
$$K^{Terr} = \frac{a}{4\pi} \sum_{n=N'+1}^{N_{\max}} \frac{2n+1}{n-1} P_n(\cos\psi)$$

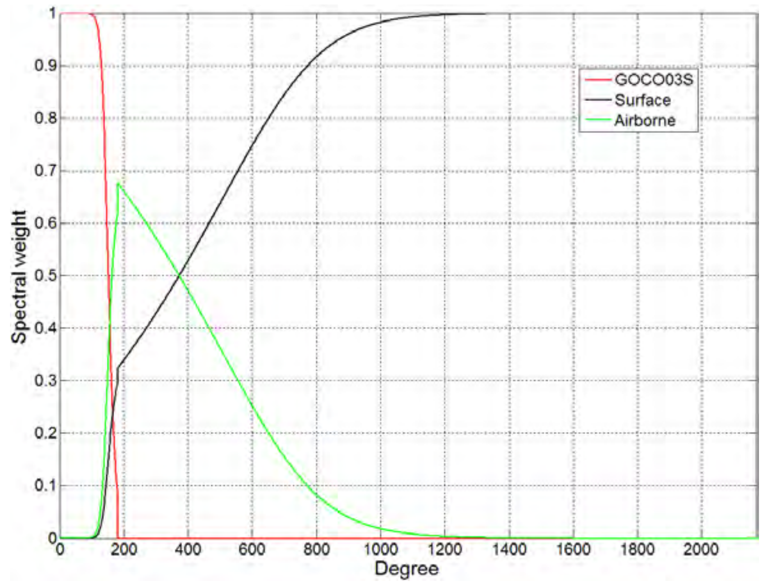
Determination of spectral weights

It is critical to have the right spectral weights for the optimal solution. The correctness of the weights depends upon the correct error degree variances. Current there are two ways to determine the weights:

1. Use of systematic and random error models (colored and white noises) in terrestrial and airborne gravity model (Agren 2004)
2. Compute the error degree covariance function from the data directly, then expand it into error degree variances (Jiang and Wang 2016)

Spectral weights for Texas

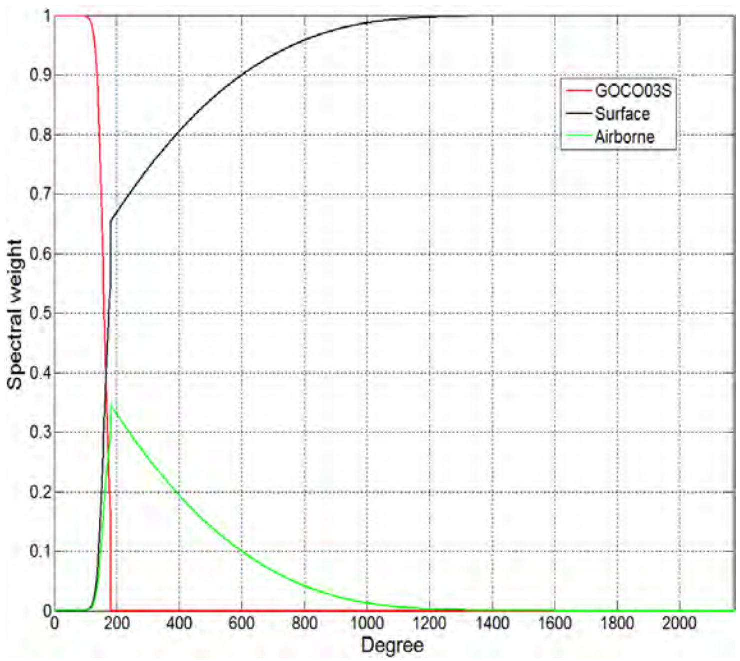




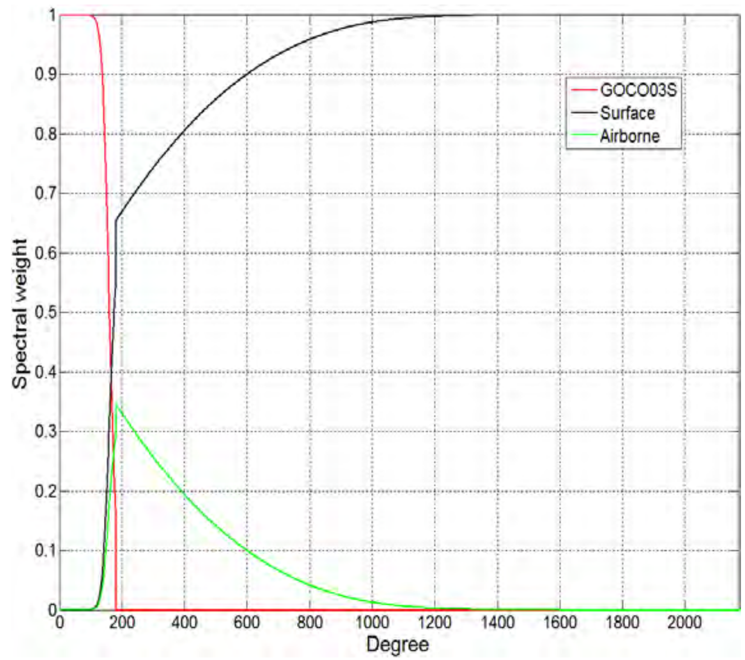
NQMAX_Air = 20

$$\sigma_0^{c,Air} = 1.5$$

Spectral weights (Texas)



NQMAX_Air = 40

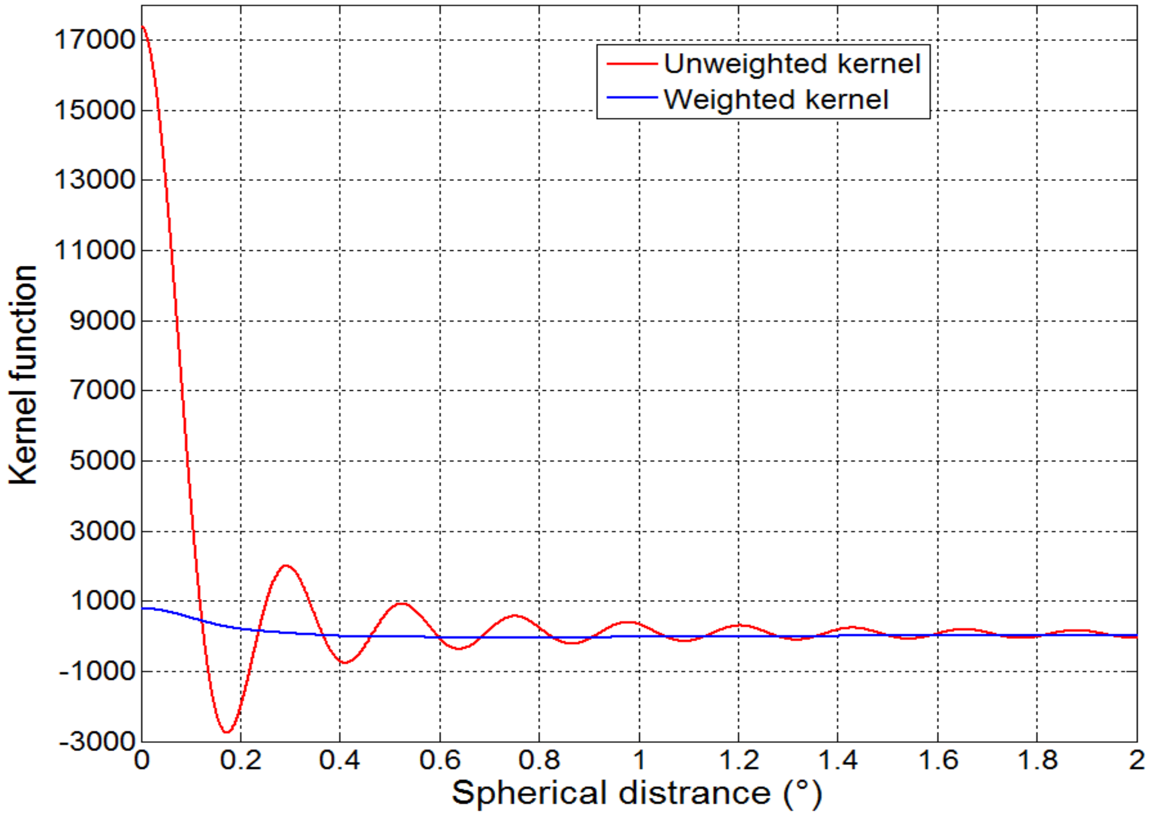


$\sigma_0^{c,Air} = 3.0$

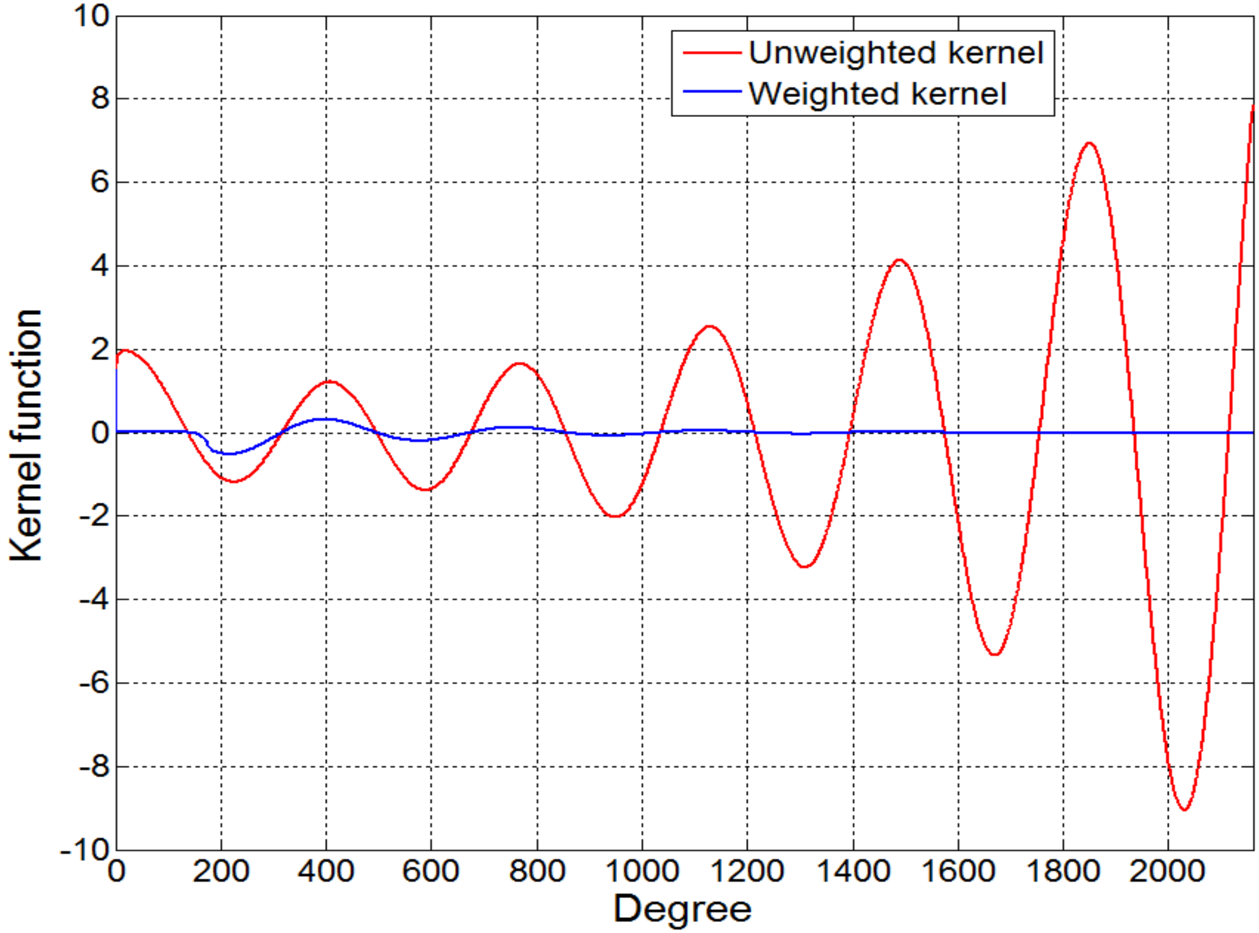
Downward Continuation Effect (Airborne)

- Downward continuation is an unstable process and may corrupt the solution if not properly treated.
- Using the spectral weights, the downward continuation is automatically stabilized. No regularization, e.g., the Tikhonov regularization is need.

Hotine's kernel function ($N_{Air} = 1600$)



Value of Hotine's kernel at $\psi = 1°$



Conclusions of Spectral Combination

Advantages:

•It weights the data according to its error characteristics and spectral contents. It makes the best combination in the spectral domain.

•The downward continuation of airborne gravity is stabilized in an optimal way. No regularization is needed.

Disadvantage:

•The error characteristics of the data can only be assessed by experience. However, it has been shown that the combined solution is not very sensitive to the weights at higher frequencies. Weights by error models and those computed from gravity data give very close results.

Research Topics of Geoid Determination

- Application of topographic reduction to airborne gravimetry
- Estimation of spectral weights for satellite model, airborne and terrestrial gravity data
- Study the spectral weights for flat, moderate and rough terrain areas. What are their general and special features?
- Topographic effect on geoid determination
- Combination of the topographic potential into gravity field modeling
- Effect of density anomaly on the mean gravity and geoid determination
- Application of ultra-high ellipsoidal harmonics for gravity field modeling