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A matrix representation of the potential second-rank gradient tensor for local modelling

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Summary. This paper introduces a new matrix computational approach to the local determination of gravity gradients, convenient for comparing with gradient signals from moving base gradiometer systems or calculating topographic effects at instrument heights. The method represents a practical alternative to the more conventional spherical harmonics formulation, primarily global in nature, and it may be considered as an extension to other previously used local representations, such as point masses. Important characteristics of the analytical development outlined herein are its conceptual simplicity and the possibility of obtaining at once, up to a certain order n, and in an arbitrary Cartesian coordinate system, the symmetric point gradient tensor of second rank.

1 Introduction

The determination of the fine structure of the Earth's gravity field still remains one of the most challenging problems of geodesy and geophysics. The repercussions which improved knowledge of the geopotential would have on many practical civilian and military applications are difficult to overstate. Satellite navigation and altimetry, inertial navigation, Earth and ocean physics, and important economic surveys, such as global assessments of petroleum and mineral resources, are but a few of the areas promising great advancement, if knowledge of high resolution gravitational disturbances should become available.

Currently, the most detailed Earth gravitational models are given through the coefficients of spherical harmonic expansions up to degree and order 180 (Rapp 1981; Lerch *et al.* 1981). Consequently, resolution of horizontal features of approximately $1^\circ \times 1^\circ$ (i.e. block size of 110 km) has been achieved. While important applications of earth models such as the ones mentioned above should be recognized (Tscherning 1983), classical techniques cannot provide an imminent extension of these series or appreciable improvements in our knowledge of the shorter wavelength gravity field variations. New technology and methods must be introduced to replace conventional ones. A significant effort is now centred around the future Geopotential Research Mission (GRM) expected to fly in 1992, and the development and implementation of new highly sophisticated hardware, such as superconducting gravity gradiometers, probably to follow on the GRM.

Spherical harmonics are ideally suited to depict the long-wavelength components of the gravity field, and traditionally the only representation generally accepted. Several equations for determining the attraction and its partial derivatives from the basic spherical potential expansion have been published (Gulick 1970; Sitzman 1971; Pines 1973; Tscherning 1976a, b; Hotine 1969, p. 181). However, all the algorithms devised to implement these equations numerically consume a significant amount of processor time, when gravity gradients affected by local disturbances (primarily embedded in the high-degree coefficients) are required. In this context, the reader may consider recently updated computer programs with more efficient recursive algorithms discussed by Tscherning, Rapp & Goad (1983). Finally, another argument frequently mentioned against a global spherical harmonics representation is our current poor understanding of the exact signal-to-noise ratio characterizing these high-degree potential coefficients.

Consequently, alternative methods for local representation of the gravity field have been introduced to avoid some of the weaknesses of the spherical harmonics approach. Some representative examples are sampling functions (Giacaglia & Lunquist 1971), point masses (Weightman 1967; Needham 1970), surface densities (Kock 1970; Morrison 1971) and finite elements (Junkins 1976; Meissl 1981).

Although abundant literature is available on the topic of possible improvements of the Earth's gravity field (surface gravity anomalies, etc.) from assumed gradiometry observations (e.g. Rummel 1979; Jekeli 1983); nevertheless, only a few authors have discussed specifically the contribution of local gravity disturbances to all the components of the gravity gradient. Besides the classical formulation used in the reduction of terrestrial torsion balance measurements extensively referenced in Mueller (1964) and the modern use of spherical harmonics (Chovitz, Lucas & Morrison 1973) or point masses for satellite gradiometry (Reed 1973) passing through the recent applications of least-squares collocation techniques in kinematical geodesy (Moritz 1971; Groten 1979; Hein 1981), very little emphasis has been given to the practical modelling of the local second-rank gradient tensor at points of arbitrary height. At a recent workshop sponsored by the NASA Geodynamics Branch, among several problem areas in gravity gradiometry, data processing was expressly mentioned and the following recommendation made: 'Simulate the analysis of gravity gradiometer data to validate processing strategies' (Wells 1984). This paper may be considered to be a contribution along these lines.

2 General background

In Soler (1984) the matrix expansion of the gravitational attraction at any exterior point $P(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ of a body of mass M was given by

$$\{f\} = G \sum_{n=0}^{\infty} \sum_{k=0}^{\langle n/2 \rangle} \frac{T_{nk}}{\bar{r}^{2(n-k)+1}} \left\{ \frac{2(k-n)-1}{\bar{r}^2} (\{\bar{x}\}^t \{\mathscr{K}\}_{nk}) \{\bar{x}\} + (n-2k)\{\mathscr{K}\}_{nk} \right\} \tag{2.1}$$

where G is the constant of gravitation and $\bar{r}$ is the magnitude of the radius vector of P. In general, the symbol $\langle \nu \rangle$ is used throughout this paper to denote the largest integer $\leqslant \nu$ and

$$T_{nk} = (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} \tag{2.2}$$

$$\{\mathscr{K}\}_{nk} = \int_M (\{\bar{x}\}^t\{x\})^{n-2k-1}(\{x\}^t\,\{x\})^k\{x\}\,dm$$

$$= \{\bar{x}\}^t \overset{\langle(n-1)/2\rangle-k}{\cdots\cdots} \{\bar{x}\}^t\,\mathscr{C}^k\,[\text{inertia tensor of rank } n]\,\{\bar{x}\} \overset{\langle n/2\rangle-k}{\cdots\cdots} \{\bar{x}\}. \tag{2.3}$$

Equation (2.1) represents in simplified matrix notation the three components of the gravitational attraction at a point P with coordinates $\{\bar{x}\}$ referred to an arbitrary right-handed Cartesian system (x_1, x_2, x_3) with origin at Ox (see Fig. 1). The three coordinates of the body's centre of mass (CM) with respect to this frame are the components of the column matrix $\{\xi\}$. Any point mass dm in the body will have coordinates $\{x\}$. The central principal axes are denoted by x_{op_i}, $i = 1, 2, 3$.

The more general nomenclature for tensors of inertia suggested by Hotine (1969, p. 165) is adopted in this work, thereby departing from the conventional terminology presented in most references, which is usually restricted only to second-rank tensors.

In matrix notation, inertia tensors of second, third and fourth rank will be defined here respectively by

$$[I] = \int_M \{x\}\,\{x\}^t\,dm = \int_M [J]\,dm \tag{2.4}$$

$$\{[I]\} = \int_M \{x\}\otimes[J]\,dm = \int_M \begin{Bmatrix} x_1[J] \\ x_2[J] \\ x_3[J] \end{Bmatrix} dm = \int_M \begin{Bmatrix} [J]_1 \\ [J]_2 \\ [J]_3 \end{Bmatrix} dm = \int_M \{[J]\}\,dm = \begin{Bmatrix} [I]_1 \\ [I]_2 \\ [I]_3 \end{Bmatrix} \tag{2.5}$$

and

$$[[I]] = \int_M \{x\}\otimes\{[J]\}^t\,dm = \int_M [[J]]\,dm = \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ \text{sym} & & [I]_{33} \end{bmatrix}. \tag{2.6}$$

The extension to higher rank inertia tensors, if desired, is easy to visualize. Consult Soler (1984) for the explicit form of the above tensors as a function of the inertial integrals (MacMillan 1930, p. 89) and the rules for transforming them under rotations.

Here, in general, every inertia tensor of odd rank is a 3×1 vector, the components of which are three matrices, called 'clusters'. Similarly, inertia tensors of even rank are 3×3 symmetric matrices of nine clusters. The contraction $\mathscr{C}$ of a tensor of rank n is another tensor of rank $n - 2$. The contraction of inertia tensors of even rank is equal to the sum of the three clusters forming the diagonal of the tensor. The contraction of inertia tensor of odd rank is equal to the contraction of each of the three components. A tensor of rank n may have a total of k successive contractions $\mathscr{C}^k$, where $k = \langle n/2\rangle$. When the clusters are matrices of 3×3 elements, then $\mathscr{C}$ and the standard symbol Tr (trace) are interchangeable. The contraction of order zero by definition is the identity contraction (e.g. $\mathscr{C}^0[I] = [I]$).

As an example, one can apply contractions to the three inertia tensors described above, as follows:

$$\mathscr{C}[I] = \mathrm{Tr}[I] = I_{11} + I_{22} + I_{33} \tag{2.7}$$

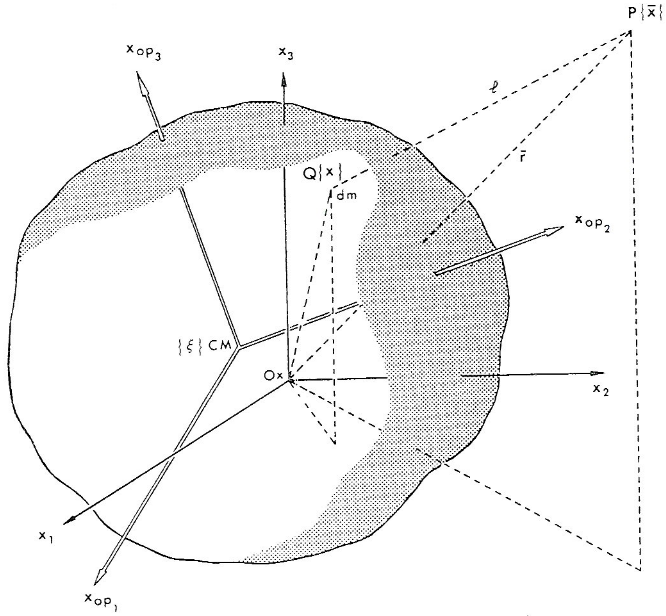

Figure 1. Coordinate systems notation and configuration.

$$\mathscr{C}\{[I]\} = \begin{Bmatrix} \mathrm{Tr}[I]_1 \\ \mathrm{Tr}[I]_2 \\ \mathrm{Tr}[I]_3 \end{Bmatrix} = \begin{Bmatrix} I_{111} + I_{221} + I_{331} \\ I_{112} + I_{222} + I_{332} \\ I_{113} + I_{223} + I_{333} \end{Bmatrix} \tag{2.8}$$

$$\mathscr{C}[[I]] = [I]_{11} + [I]_{22} + [I]_{33} \tag{2.9}$$

$$\mathscr{C}^2\,[[I]] = \mathscr{C}\mathscr{C}[[I]] = \mathrm{Tr}[I]_{11} + \mathrm{Tr}[I]_{22} + \mathrm{Tr}[I]_{33}. \tag{2.10}$$

Consequently, the maximum possible contraction of inertia tensors of odd and even rank are vectors and scalars respectively.

Also in Soler (1984) it was shown that

$$\{\bar{x}\}^t\{\mathscr{K}\}_{nk} = \mathscr{I}_{nk} = \int_M (\{\bar{x}\}^t\,\{x\})^{n-2k}(\{x\}^t\,\{x\})^k\,dm$$

$$= \{x\}^t \,\underset{\cdots\cdots\cdots}{\langle (n+1)/2\rangle - k}\, \{x\}^t\mathscr{C}^k\,[\text{inertia tensor of rank } n]\ \{x\}\,\underset{\cdots\cdots}{\langle n/2\rangle - k}\,\{x\}\,. \tag{2.11}$$

In equations (2.3) and (2.11) operations involving tensors of first rank (vectors) and general inertia tensors of rank n are implicit. These operations follow matrix multiplication rules, but 'component by component', similarly to the standard multiplication rules of element by element. For example,

$$\{\bar{x}\}^t[[I]]\{\bar{x}\} = \{\bar{x}\}^t \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ \text{sym} & & [I]_{33} \end{bmatrix} \{\bar{x}\}$$

$$= \bar{x}_1^2\,[I]_{11} + \bar{x}_2^2\,[I]_{22} + \ldots + 2\bar{x}_2\,\bar{x}_3\,[I]_{23}. \tag{2.12}$$

3 The gradient matrix of the gravitational potential V

It is well known that the gradient matrix (gradient tensor of second rank) of the potential V at P in an irrotational vector field can be expressed in any of the following forms (recall that the gradient of a tensor of rank n is another tensor of rank $n+1$):

$$\left\{\frac{\partial}{\partial \bar{x}}\right\}\underline{\left\{\frac{\partial V}{\partial \bar{x}}\right\}^t} = \left\{\frac{\partial}{\partial \bar{x}}\right\}\underline{\{f\}^t} = \begin{bmatrix} \dfrac{\partial^2 V}{\partial \bar{x}_1^2} & \dfrac{\partial^2 V}{\partial \bar{x}_1 \partial \bar{x}_2} & \dfrac{\partial^2 V}{\partial \bar{x}_1 \partial \bar{x}_3} \\ & \dfrac{\partial^2 V}{\partial \bar{x}_2^2} & \dfrac{\partial^2 V}{\partial \bar{x}_2 \partial \bar{x}_3} \\ \text{sym} & & \dfrac{\partial^2 V}{\partial \bar{x}_3^2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial \bar{x}_1} & \dfrac{\partial f_2}{\partial \bar{x}_1} & \dfrac{\partial f_3}{\partial \bar{x}_1} \\ & \dfrac{\partial f_2}{\partial \bar{x}_2} & \dfrac{\partial f_3}{\partial \bar{x}_2} \\ \text{sym} & & \dfrac{\partial f_3}{\partial \bar{x}_3} \end{bmatrix} \tag{3.1}$$

The partial vector operator $\{\partial/\partial \bar{x}\}$ shown above always applies to the quantities to its right contained over the horizontal bar.

Recalling from (2.11) that $\{\bar{x}\}^t\{\mathscr{K}\}_{nk} = \mathscr{I}_{nk}$, the transpose of $\{f\}$ is readily obtained from (2.1)

$$\{f\}^t = G \sum_{n=0}^{\infty} \sum_{k=0}^{\langle n/2 \rangle} \frac{T_{nk}}{\bar{r}^{2(n-k)+1}} \left\{ \frac{2(k-n)-1}{\bar{r}^2} \mathscr{I}_{nk} \{\bar{x}\}^t + (n-2k)\{\mathscr{K}\}_{nk}^t \right\}. \tag{3.2}$$

Then after taking partial derivatives with respect to the quantities depending on $\{\bar{x}\}$ in equation (3.2), it is not difficult to show

$$\begin{aligned} \left\{\frac{\partial}{\partial \bar{x}}\right\}\underline{\{f\}^t} = G \sum_{n=0}^{\infty} \sum_{k=0}^{\langle n/2 \rangle} & \frac{2(k-n)-1}{\bar{r}^{2(n-k)+3}} T_{nk} \\ \times \Bigg[& \frac{2(k-n)-3}{\bar{r}^2} \mathscr{I}_{nk}\{\bar{x}\}\{\bar{x}\}^t + \left\{\frac{\partial}{\partial \bar{x}}\right\} \underline{\mathscr{I}_{nk}\{\bar{x}\}^t} \\ & + (n-2k)\left[\{\bar{x}\}\{\mathscr{K}\}_{nk}^t + \frac{\bar{r}^2}{2(k-n)-1}\left\{\frac{\partial}{\partial \bar{x}}\right\}\underline{\{\mathscr{K}\}_{nk}^t}\right]\Bigg]. \end{aligned} \tag{3.3}$$

Further, by recognizing that

$$\left\{\frac{\partial}{\partial \bar{x}}\right\}\underline{\{\bar{x}\}^t} = [1] \tag{3.4}$$

where $[1]$ is the 3×3 unit matrix, one immediately obtains

$$\left\{\frac{\partial}{\partial \bar{x}}\right\}\underline{\mathscr{I}_{nk}\{\bar{x}\}^t} = \left\{\frac{\partial}{\partial \bar{x}}\right\}\underline{\mathscr{I}_{nk}}\{\bar{x}\}^t + \mathscr{I}_{nk}[1]. \tag{3.5}$$

Moreover, from (2.11) and (2.3) it follows

$$\left\{\frac{\partial}{\partial \bar{x}}\right\}\underline{\mathscr{I}_{nk}} = (n-2k)\int_M (\{\bar{x}\}^t\{x\})^{n-2k-1}(\{x\}^t\{x\})^k\{x\}\,dm = (n-2k)\{\mathscr{K}\}_{nk}. \tag{3.6}$$

Thus, replacing the above in (3.3), after grouping terms and simplifying, one finally arrives at

$$\left\{\frac{\partial}{\partial \bar{x}}\right\} \underline{\{f\}^t} = G \sum_{n=0}^{\infty} \sum_{k=0}^{\langle n/2 \rangle} \frac{2(k-n)-1}{\bar{r}^{2(n-k)+3}} T_{nk} \times \left[\!\left[\mathscr{I}_{nk} \left[[1] + \frac{2(k-n)-3}{\bar{r}^2} [\bar{J}] \right] + (n-2k) \left[\{\bar{x}\} \{\mathscr{K}\}^t_{nk} + \{\mathscr{K}\}_{nk} \{\bar{x}\}^t + \frac{\bar{r}^2}{2(k-n)-1} \left\{\frac{\partial}{\partial \bar{x}}\right\} \underline{\{\mathscr{K}\}^t_{nk}} \right] \right]\!\right] \tag{3.7}$$

where the substitution $\{\bar{x}\}\{\bar{x}\}^t = [\bar{J}]$ was implemented.

Equation (3.7) gives the full second rank gradient tensor of V as a function of quantities already known, such as the scalar $\mathscr{I}_{nk}$ and the vector $\{\mathscr{K}\}_{nk}$. Still to be explicitly defined is the matrix $\{\partial/\partial\bar{x}\}\{\mathscr{K}\}^t_{nk}$ (i.e. the gradient of the vector $\{\mathscr{K}\}_{nk}$.)

The gradient of $\{\mathscr{K}\}_{nk}$ may be computed, first making use of (2.3)

$$\{\mathscr{K}\}^t_{nk} = \int_M (\{\bar{x}\}^t\{x\})^{n-2k-1}(\{x\}^t\{x\})^k\{x\}^t \, dm \tag{3.8}$$

and then,

$$\left\{\frac{\partial}{\partial \bar{x}}\right\} \{\mathscr{K}\}^t_{nk} = (n-2k-1) \int_M (\{\bar{x}\}^t\{x\})^{n-2k-2}(\{x\}^t\{x\})^k [J] \, dm$$

$$= (n-2k-1)[\partial \mathscr{K}]_{nk} \tag{3.9}$$

where (see Appendix A)

$$[\partial \mathscr{K}]_{nk} = \underbrace{\{\bar{x}\}^t \cdots \{\bar{x}\}^t}_{\langle (n-1)/2 \rangle - k} \mathscr{C}^k [\text{inertia tensor of rank } n] \underbrace{\{\bar{x}\} \cdots \{\bar{x}\}}_{\langle n/2 \rangle - (k+1)}. \tag{3.10}$$

Notice that the matrix $\{\partial/\partial\bar{x}\}\{\mathscr{K}\}^t_{nk}$ is a 3×3 zero matrix when $n = 2k+1$. Moreover its substitution in (3.7) is not required when $n = 2k$; therefore equations (3.9) and (3.10) need to be defined only for values of $k \neq \langle n/2 \rangle$.

4 The trace of the potential second-rank gradient tensor

In this section, primarily as a check of equation (3.7), Laplace's condition will be proved.

In matrix notation, this condition can be written

$$\mathscr{C}\left[\left\{\frac{\partial}{\partial \bar{x}}\right\}\underline{\left\{\frac{\partial V}{\partial \bar{x}}\right\}^t}\right] = \mathscr{C}\left[\left\{\frac{\partial}{\partial \bar{x}}\right\} \underline{\{f\}^t}\right] = 0. \tag{4.1}$$

Recalling that

$$\mathscr{C}[\{\bar{x}\}\{\mathscr{K}\}^t_{nk}] = \mathscr{C}[\{\mathscr{K}\}_{nk}\{\bar{x}\}^t] = \{\bar{x}\}^t\{\mathscr{K}\}_{nk} = \mathscr{I}_{nk} \tag{4.2}$$

and, from (3.9) and (2.11)

$$\mathscr{C}\left[\left\{\frac{\partial}{\partial \bar{x}}\right\} \underline{\{\mathscr{K}\}^t_{nk}}\right] = (n-2k-1) \int_M (\{\bar{x}\}^t\{x\})^{n-2k-2}(\{x\}^t\{x\})^{k+1} \, dm$$

$$= (n-2k-1)\,\mathscr{I}_{n,k+1}. \tag{4.3}$$

Consequently,

$$\mathscr{I}_{n,\,k+1} = \mathscr{C}\,[\partial\,\mathscr{K}]_{nk}. \tag{4.4}$$

Therefore, taking contractions in equation (3.7) and substituting the equalities given above, one arrives at

$$\mathscr{C}\left[\left\{\frac{\partial}{\partial\bar{x}}\right\}\underline{\{f\}^t}\right] = G\sum_{n=0}^{\infty}\sum_{k=0}^{\langle n/2\rangle}\frac{2(k-n)-1}{\bar{r}^{2(n-k)+3}}\,T_{nk}\left(-2k\,\mathscr{I}_{nk} + \frac{(n-2k)(n-2k-1)}{2(k-n)-1}\,\bar{r}^2\mathscr{I}_{n,\,k+1}\right)$$

$$= G\sum_{n=0}^{\infty}\sum_{k=0}^{\langle n/2\rangle}(A_{nk}+B_{nk}) \tag{4.5}$$

where

$$A_{nk} = -\,2k\,\frac{2(k-n)-1}{\bar{r}^{2(n-k)+3}}\,T_{nk}\,\mathscr{I}_{nk} \tag{4.6a}$$

and

$$B_{nk} = \frac{(n-2k)(n-2k-1)}{\bar{r}^{2(n-k)+1}}\,T_{nk}\,\mathscr{I}_{n,\,k+1}. \tag{4.6b}$$

By simple inspection, it follows immediately that in general

$$A_{n,0} = B_{n,\langle n/2\rangle} = 0 \qquad \text{for any } n. \tag{4.7}$$

Therefore, explicitly

$A_{00}=0;\;\; A_{10}=0;\;\; A_{20}=0;\;\; A_{21};\;\; A_{30}=0;\;\; A_{31};\;\; A_{40}=0;\;\; A_{41};\;\; A_{42};\;\; \ldots$

$B_{00}=0;\;\; B_{10}=0;\;\; B_{20};\;\; B_{21}=0;\;\; B_{30};\;\; B_{31}=0;\;\; B_{40};\;\; B_{41};\;\; B_{42}=0;\;\; \ldots$

and equation (4.5) can be rewritten

$$\mathscr{C}\left[\left\{\frac{\partial}{\partial\bar{x}}\right\}\underline{\{f\}^t}\right] = G\sum_{n=2}^{\infty}\sum_{k=0}^{\langle n/2\rangle-1}(A_{n,\,k+1}+B_{nk}). \tag{4.8}$$

Thus, substituting the corresponding values of $A_{n,\,k+1}$ and B_{nk} from equations (4.6) above, one has

$$\mathscr{C}\left[\left\{\frac{\partial}{\partial\bar{x}}\right\}\underline{\{f\}^t}\right] = G\sum_{n=2}^{\infty}\sum_{k=0}^{\langle n/2\rangle-1}\left(-\,2(k+1)\,\frac{2(k-n)+1}{\bar{r}^{2(n-k)+1}}\,T_{n,\,k+1}\mathscr{I}_{n,\,k+1}\right.$$

$$\left. + \frac{(n-2k)(n-2k-1)}{\bar{r}^{2(n-k)+1}}\,T_{nk}\,\mathscr{I}_{n,\,k+1}\right)$$

$$= G\sum_{n=2}^{\infty}\sum_{k=0}^{\langle n/2\rangle-1}\frac{\mathscr{I}_{n,\,k+1}}{\bar{r}^{2(n-k)+1}}$$

$$\times\left(-\,2(k+1)\,(2(k-n)+1)\;T_{n,\,k+1} + (n-2k)(n-2k-1)\;T_{nk}\right) \tag{4.9}$$

but from Kaula (1966, p. 6) it is known that

$$T_{n,k+1} = -\frac{(n-2k)(n-2k-1)}{2(k+1)(2(n-k)-1)} T_{nk}. \tag{4.10}$$

Therefore, after inserting (4.10) into (4.9) it is finally shown that equation (4.1) holds.

This proof of Laplace's equation, using matrix notation, although obviously not the most direct (see MacMillan 1930, p. 32), contrasts with the mathematical complexity required by other recently suggested subindex notations (Grafarend 1980; Kleusberg 1984).

5 First terms of the expansion of the gravitational potential second rank gradient tensor

5.1 ZERO-ORDER TERM

From equation (3.7) and for $n = 0$ and $k = 0$, it can be shown that

$$\left\{\frac{\partial}{\partial \bar{x}}\right\} \underline{\{f\}_0^t} = -\frac{GM}{\bar{r}^3} \left[[1] - \frac{3}{\bar{r}^2}[\bar{J}]\right]. \tag{5.1}$$

Use of this term alone assumes a point mass situated at the coordinate origin (see Hopkins 1973; Reed 1973; Jordan 1978).

5.2 FIRST-ORDER TERM

For $n = 1, k = 0$

$$\left\{\frac{\partial}{\partial \bar{x}}\right\} \underline{\{f\}_1^t} = -\frac{3GM}{\bar{r}^5} \left[\!\left[(\{\bar{x}\}^t\{\xi\})\left[[1] - \frac{5}{\bar{r}^2}[\bar{J}]\right] + \{\bar{x}\}\{\xi\}^t + \{\xi\}\{\bar{x}\}^t \right]\!\right]. \tag{5.2}$$

Consequently, as expected, if we choose the origin of the reference coordinate system at the CM of the body, there is no contribution from the first-order term to the gradient matrix of V.

5.3 SECOND-ORDER TERM

In this case $n = 2$ and k takes the values $k = 0, 1$. Therefore, from equation (3.7) after simple matrix manipulation and simplifications,

$$\begin{aligned}\left\{\frac{\partial}{\partial \bar{x}}\right\} \underline{\{f\}_2^t} = &-\frac{15G}{2\bar{r}^7} \left[\!\left[(\{\bar{x}\}^t[I]\{\bar{x}\})\left[[1] - \frac{7}{\bar{r}^2}[\bar{J}]\right] + 2[\bar{J}][I] + 2[I][\bar{J}] \right]\!\right] \\ &+ \frac{3G}{2\bar{r}^5}\left[\!\left[\mathscr{C}[I]\left[[1] - \frac{5}{\bar{r}^2}[\bar{J}]\right] + 2[I] \right]\!\right].\end{aligned} \tag{5.3}$$

5.4 THIRD-ORDER TERM

Now $n = 3$ and $k = 0, 1$. Thus

$$\begin{aligned}\left\{\frac{\partial}{\partial \bar{x}}\right\} \underline{\{f\}_3^t} = -\frac{35G}{2\bar{r}^9} &\left[\!\left[(\{\bar{x}\}^t[\{\bar{x}\}^t\{[I]\}]\{\bar{x}\})\left[[1] - \frac{9}{\bar{r}^2}[\bar{J}]\right] \right.\right. \\ &\left.\left. + 3[\bar{J}][\{\bar{x}\}^t\{[I]\}] + 3[\{\bar{x}\}^t\{[I]\}][\bar{J}] \right]\!\right]\end{aligned}$$

$$+\frac{15G}{2\bar{r}^7}\left[\!\!\left[(\{\bar{x}\}^t\mathscr{C}\{[I]\})\left[[1]-\frac{7}{\bar{r}^2}[\bar{J}]\right]+\mathscr{C}\{[I]\}\{\bar{x}\}^t\right.\right.$$

$$\left.\left.+\{\bar{x}\}\{\mathscr{C}\{[I]\}\}^t+2\{\bar{x}\}^t\{[I]\}\right]\!\!\right]. \tag{5.4}$$

5.5 FOURTH-ORDER TERM

Limiting the explicit matrix expansion to the fourth-order term, now $n=4$ and $k=0, 1, 2$

$$\left\{\frac{\partial}{\partial\bar{x}}\right\}\underline{\{f\}_4^t}=\frac{315G}{8\bar{r}^{11}}\left[\!\!\left[[\{\bar{x}\}^t[\{\bar{x}\}^t[[I]]\{\bar{x}\}]\{\bar{x}\}]\left[[1]-\frac{11}{\bar{r}^2}[\bar{J}]\right]\right.\right.$$

$$\left.\left.+4[\{\bar{x}\}^t[[I]]\{\bar{x}\}][\bar{J}]+4[\bar{J}][\{\bar{x}\}^t[[I]]\{\bar{x}^t\}]\right]\!\!\right]$$

$$+\frac{105G}{4\bar{r}^9}\left[\!\!\left[(\{\bar{x}\}^t\mathscr{C}[[I]]\{\bar{x}\})\left[[1]-\frac{9}{\bar{r}^2}[\bar{J}]\right]+2[\{\bar{x}\}^t[[I]]\{\bar{x}\}]\right.\right.$$

$$\left.\left.+2\mathscr{C}[[I]][\bar{J}]+2[\bar{J}]\mathscr{C}[[I]]\right]\!\!\right]$$

$$-\frac{15G}{8\bar{r}^7}\left[\!\!\left[\mathscr{C}^2[[I]]\left[[1]-\frac{7}{\bar{r}^2}[\bar{J}]\right]+4\mathscr{C}[[I]]\right]\!\!\right]. \tag{5.5}$$

6 Final general equation

A simplified general matrix equation valid for $n \geqslant 0$ can be written to replace (3.7) defining $\{\bar{x}\}^{-1} \doteq \bar{r}^{-2}\{\bar{x}\}^t$ in order to accommodate the case $n=1$

$$\left\{\frac{\partial}{\partial\bar{x}}\right\}\underline{\{f\}_n^t}=G\sum_{n=0}^{\infty}\sum_{k=0}^{\langle n/2\rangle}\frac{2(k-n)-1}{\bar{r}^{2(n-k)+3}}T_{nk}$$

$$\times\left[\!\!\left[\mathscr{I}_{nk}\left[[1]+\frac{2(k-n)-3}{\bar{r}^2}[\bar{J}]\right]+2k[\partial\mathscr{K}]_{n,k-1}\right.\right.$$

$$\left.\left.+(n-2k)[\![[\partial\mathscr{K}]_{nk}[\bar{J}]+[[\partial\mathscr{K}]_{nk}[\bar{J}]]^t]\!]\right]\!\!\right]. \tag{6.1}$$

Notice that in practical applications one only needs to compute the matrix $[\partial\mathscr{K}]_{nk}$ since

$$\mathscr{I}_{nk}=\{\bar{x}\}^t[\partial\mathscr{K}]_{nk}\{\bar{x}\} \quad \text{when } n\neq 2k \quad \text{or} \quad n\neq 2k+1 \quad (\text{i.e. } k\neq\langle n/2\rangle) \tag{6.2}$$

and

$$\mathscr{I}_{nk}=\mathscr{C}[\partial\mathscr{K}]_{n,k-1} \quad \text{when } n=2k \quad \text{or} \quad n=2k+1 \quad (\text{i.e. } k=\langle n/2\rangle). \tag{6.3}$$

Consequently

$$\mathscr{C}[[\partial\mathscr{K}]_{nk}[\bar{J}]]=\{\bar{x}\}^t[\partial\mathscr{K}]_{nk}\{\bar{x}\}=\mathscr{I}_{nk}. \tag{6.4}$$

Taking traces of (6.1), and using the above equations, Laplace's condition immediately follows.

As an illustrative example, and in order to facilitate the understanding of the reader, below are given as a function of the inertia tensors, the 3×3 matrices $[\partial \mathscr{K}]_{nk}$ required in the expansion to order $n = 7$ of the second-rank gradient tensor.

$n = 2$

$$[\partial \mathscr{K}]_{2,0} = [I] \tag{6.5}$$

$n = 3$

$$[\partial \mathscr{K}]_{3,0} = \{\bar{x}\}^t \{[I]\} \tag{6.6}$$

$n = 4$

$$[\partial \mathscr{K}]_{4,0} = \{\bar{x}\}^t [[I]] \{\bar{x}\} \tag{6.7}$$

$$[\partial \mathscr{K}]_{4,1} = \mathscr{C}[[I]] \tag{6.8}$$

$n = 5$

$$[\partial \mathscr{K}]_{5,0} = \{\bar{x}\}^t [\{\bar{x}\}^t \{[[I]]\}] \{\bar{x}\} \tag{6.9}$$

$$[\partial \mathscr{K}]_{5,1} = \{\bar{x}\}^t \mathscr{C} \{[[I]]\} \tag{6.10}$$

$n = 6$

$$[\partial \mathscr{K}]_{6,0} = \{\bar{x}\}^t [\{\bar{x}\}^t [[[I]]] \{\bar{x}\}] \{\bar{x}\} \tag{6.11}$$

$$[\partial \mathscr{K}]_{6,1} = \{\bar{x}\}^t \mathscr{C} [[[I]]] \{\bar{x}\} \tag{6.12}$$

$$[\partial \mathscr{K}]_{6,2} = \mathscr{C}^2 [[[I]]] \tag{6.13}$$

$n = 7$

$$[\partial \mathscr{K}]_{7,0} = \{\bar{x}\}^t [\{\bar{x}\}^t [\{\bar{x}\}^t \{[[[I]]]\}] \{\bar{x}\}] \{\bar{x}\} \tag{6.14}$$

$$[\partial \mathscr{K}]_{7,1} = \{\bar{x}\}^t [\{\bar{x}\}^t \mathscr{C} \{[[[I]]]\}] \{\bar{x}\} \tag{6.15}$$

$$[\partial \mathscr{K}]_{7,2} = \{\bar{x}\}^t \mathscr{C}^2 \{[[[I]]]\}. \tag{6.16}$$

The general expression (6.1) of this section in conjunction with equation (3.10) provides a novel matrix approach for computing the entire second-rank gradient tensor at any point P in space.

In practical applications the local relief can be approximated by a discrete number of finite elements (blocks) of arbitrary shape. The expansion depends upon the coordinates of P and the inertia tensors of the element referred to some Cartesian coordinate system. Incidentally, because the inertia tensors are functions of inertial integrals each element will be characterized by its size and density. The final gradient tensor at P will be equal to the sum of each individual contribution from all modelled elements.

For completeness, Appendix B recapitulates explicitly in matrix form the inertia tensors of even rank up to $n = 6$ for a single homogeneous element of parallelepipedon shape.

The selection of element sizes and the maximum order of the expansion will depend primarily on the desired resolution and the specific problem at hand (e.g. satellite versus airborne gradiometry). Assume, for example, a typical terrain element ($\rho = 2.65\,\mathrm{g\,cm^{-2}}$) of a height of 2 km, and 2 km × 2 km base (approximately an equiangular block of $1' \times 1'$). Then, the contribution of the expansion fourth order term given explicitly by equation (5.5) to the vertical gradient $\partial^2 V/\partial \bar{x}_3^2$ along the line $\bar{x}_1 = \bar{x}_2 = 0$ will be larger than 1 EU (EU = Eötvös unit = 0.1 μgal m^{-1}) when $\bar{x}_3 < 2.67$ km.

However, the primary intention of this paper was restricted to describing the theory without emphasizing any particular simulation applicable to specific problems. Thus, it is left to the reader to decide on the best possible modelling and expansion, depending on the availability of data and individual constraints.

7 Conclusions

The method proposed here introduces a new matrix formulation for post-observational processing and interpretation of gravity gradiometry measurements obtained from moving base gradiometer prototypes.

It is known that contrary to conventional gravity information, gravity gradiometry signals are very sensitive to density contrast. If the nearest surface topographic effects can be modelled with some reliability, the discrimination of certain underground geological structures may become possible in geophysical explorations. This is due primarily to the well established fact that higher derivatives provide higher resolution and are able to detect variations due to buried mass anomalies, where standard gravity surveys fail (Hammer & Anzoleaga 1975).

Generally speaking, every analysis performed to date for interpreting the direct effects of non-uniform terrain on gravity gradients at airborne heights has been limited to two-dimensional models. These essentially compute only the vertical gradient along a single profile parallel to the flight line, and disregard the influence of nearby irregular topographic masses, if any (Hammer 1976). The approach suggested in this paper, although strictly deterministic in nature, has considerable flexibility, being adaptable to any type of topography likely to be encountered in practice. The only requirements are a digitized mean elevation data base, in some arbitrary vertical datum, and a set of assumed densities (e.g. ice, seawater, crust, mantle). The sophistication of the model depends on the size of the elements involved and the reliability of the densities.

Finally, it should be mentioned that the methodology described in this work, due to its innovative matrix formulation, is most appropriate for deriving expressions involving higher-rank gradient tensors (e.g. third, fourth) and their corresponding transformations due to rotations between different coordinate systems (e.g. local, inertial, platform) at the point of observation.

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Appendix A: the gradient of the vector $\{\mathscr{K}\}_{nk}$

It was shown before in (3.9) that

$$\left\{\frac{\partial}{\partial \bar{x}}\right\}\{\mathscr{K}\}_{nk}^t = (n-2k-1)\int_M (\{\bar{x}\}^t\{x\})^{n-2k-2}(\{x\}^t\{x\})^k\{x\}\{x\}^t\,dm$$

$$= (n-2k-1)[\partial\mathscr{K}]_{nk}.$$

Let us determine the values inside the above integral, in which the exponents are positive. This occurs when $n \neq 2k$ or $n \neq 2k+1$, that is, for values of n which make $k \neq \langle n/2\rangle$. As discussed in Section 3, these are the only values of n where the 3×3 matrix $[\partial\mathscr{K}]_{nk}$ is actually required for practical applications.

Then, for $n = 2$, $k = 0$

$$\{x\}\{x\}^t = [J].$$

For $n = 3$, and $k = 0$

$$(\{\bar{x}\}^t\{x\})[J] = (\bar{x}_1x_1 + \bar{x}_2x_2 + \bar{x}_3x_3)[J] = \{\bar{x}\}^t\begin{Bmatrix} x_1[J] \\ x_2[J] \\ x_3[J]\end{Bmatrix} = \{\bar{x}\}^t\begin{Bmatrix}[J]_1 \\ [J]_2 \\ [J]_3\end{Bmatrix} = \{\bar{x}\}^t\{[J]\}.$$

For $n = 4$ and $k = 0$

$$(\{\bar{x}\}^t\{x\})^2[J] = (\{\bar{x}\}^t[J]\{\bar{x}\})[J] = (\bar{x}_1^2x_1^2 + \ldots + 2\bar{x}_2\bar{x}_3x_2x_3)[J]$$

$$= \bar{x}_1^2[J]_{11} + \ldots + 2\bar{x}_2\bar{x}_3[J]_{23} = \{\bar{x}\}^t[[J]]\{\bar{x}\}.$$

For $n = 4$ and $k = 1$

$$(\{x\}^t\{x\})[J] = (x_1^2 + x_2^2 + x_3^2)[J] = [J]_{11} + [J]_{22} + [J]_{33} = \mathscr{C}[[J]].$$

Similarly, if $n = 5$ and $k = 0$,

$$(\{\bar{x}\}^t\{x\})^3[J] = \{\bar{x}\}^t[\{\bar{x}\}^t\{[[J]]\}]\{\bar{x}\}$$

and for $n = 5$, $k = 1$

$$(\{\bar{x}\}^t\{x\})(\{x\}^t\{x\})[J] = (\{x\}^t\{x\})\{\bar{x}\}^t\{[J]\} = \{\bar{x}\}^t\mathscr{C}\{[[J]]\} \text{ etc.}$$

It then holds that in general and after integrating over the total mass of the body

$$[\partial\mathscr{K}]_{nk} = \{\bar{x}\}^t \underset{\langle(n-1)/2\rangle-k}{\cdots\cdots} \{\bar{x}\}^t\mathscr{C}^k[\text{inertia tensor of rank } n]\{\bar{x}\}\underset{\langle n/2\rangle-(k+1)}{\cdots\cdots}\{\bar{x}\}.$$

As explained before, the above equation is defined only for values of n when $k \neq \langle n/2\rangle$.

Appendix B: inertia tensors (up to rank six) of a parallelepipedon element

Let us introduce at the mass centre (CM) of a homogeneous parallelepipedon element of dimensions $2a$, $2b$, $2c$ and density ρ a Cartesian coordinate system (x_1, x_2, x_3) as shown in Fig. A1.

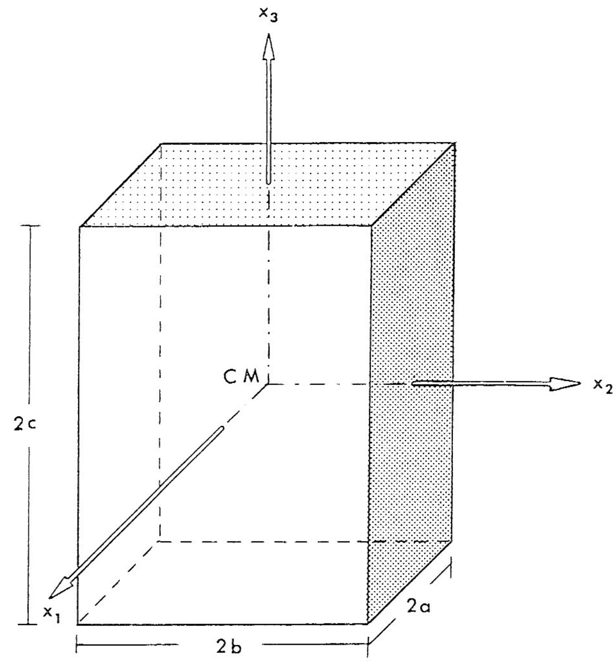

Figure A1. Coordinate system and parallelepipedon element relationships.

The inertial integrals of this particular element can be computed using the general equation given in (MacMillan 1930, p. 89), namely

$$\rho \int_V x_1^{2p} x_2^{2q} x_3^{2r}\, dx_1\, dx_2\, dx_3 = M \frac{a^{2p}\, b^{2q}\, c^{2r}}{(2p+1)(2q+1)(2r+1)}$$

where p, q and r are integers and M is the mass of the element. Clearly, because the planes x_1x_2, x_1x_3 and x_2x_3 are planes of symmetry, the exponents of the variables x_1, x_2 and x_3 inside the integral will always be even numbers.

The matrix form of the non-zero inertia tensors (up to rank six) corresponding to the element of Fig. A1 are given below.

Inertia tensor of rank zero: it is very well known that the inertia tensor of rank zero is the scalar M.

Inertia tensor of rank two:

$$[I] = \frac{M}{3} \begin{bmatrix} a^2 & 0 & 0 \\ & b^2 & 0 \\ \text{sym} & & c^2 \end{bmatrix}.$$

Inertia tensor of rank four $[[I]]$:

$$[[I]] = \frac{M}{9}\begin{bmatrix} a^2\begin{bmatrix} \frac{9}{5}a^2 & 0 & 0 \\ & b^2 & 0 \\ \text{sym} & & c^2 \end{bmatrix} & a^2\begin{bmatrix} 0 & b^2 & 0 \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix} & a^2\begin{bmatrix} 0 & 0 & c^2 \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix} \\ & b^2\begin{bmatrix} a^2 & 0 & 0 \\ & \frac{9}{5}b^2 & 0 \\ \text{sym} & & c^2 \end{bmatrix} & b^2\begin{bmatrix} 0 & 0 & 0 \\ & 0 & c^2 \\ \text{sym} & & 0 \end{bmatrix} \\ \text{sym} & & c^2\begin{bmatrix} a^2 & 0 & 0 \\ & b^2 & 0 \\ \text{sym} & & \frac{9}{5}c^2 \end{bmatrix} \end{bmatrix}$$

Inertia tensor of rank six $[[[I]]]$:

The inertia tensor of rank six may be written,

$$[[[I]]] = \begin{bmatrix} [[I]]_{11} & [[I]]_{12} & [[I]]_{13} \\ & [[I]]_{22} & [[I]]_{23} \\ \text{sym} & & [[I]]_{33} \end{bmatrix}$$

where the diagonal clusters are given by

$$[[I]]_{11} = \frac{M}{15}a^2\begin{bmatrix} a^2\begin{bmatrix} \frac{15}{7}a^2 & 0 & 0 \\ & b^2 & 0 \\ \text{sym} & & c^2 \end{bmatrix} & a^2\begin{bmatrix} 0 & b^2 & 0 \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix} & a^2\begin{bmatrix} 0 & 0 & c^2 \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix} \\ & b^2\begin{bmatrix} a^2 & 0 & 0 \\ & b^2 & 0 \\ \text{sym} & & \frac{5}{9}c^2 \end{bmatrix} & b^2\begin{bmatrix} 0 & 0 & 0 \\ & 0 & \frac{5}{9}c^2 \\ \text{sym} & & 0 \end{bmatrix} \\ \text{sym} & & c^2\begin{bmatrix} a^2 & 0 & 0 \\ & \frac{5}{9}b^2 & 0 \\ \text{sym} & & c^2 \end{bmatrix} \end{bmatrix}$$

$$
= \begin{bmatrix} [\mathfrak{A}] & [\mathfrak{B}] & [\mathfrak{D}] \\ & [\mathfrak{C}] & [\mathfrak{E}] \\ \text{sym} & & [\mathfrak{F}] \end{bmatrix}
$$

$$
[[I]]_{22} = \frac{M}{15} b^2 \begin{bmatrix}
a^2 \begin{bmatrix} a^2 & 0 & 0 \\ & b^2 & 0 \\ \text{sym} & & \frac{5}{9} c^2 \end{bmatrix} & a^2 \begin{bmatrix} 0 & b^2 & 0 \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix} & a^2 \begin{bmatrix} 0 & 0 & \frac{5}{9} c^2 \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix} \\
& b^2 \begin{bmatrix} a^2 & 0 & 0 \\ & \frac{15}{7} b^2 & 0 \\ \text{sym} & & c^2 \end{bmatrix} & b^2 \begin{bmatrix} 0 & 0 & 0 \\ & 0 & c^2 \\ \text{sym} & & 0 \end{bmatrix} \\
\text{sym} & & c^2 \begin{bmatrix} \frac{5}{9} a^2 & 0 & 0 \\ & b^2 & 0 \\ \text{sym} & & c^2 \end{bmatrix}
\end{bmatrix}
$$

$$
= \begin{bmatrix} [\mathfrak{C}] & [\mathfrak{G}] & [\mathfrak{I}] \\ & [\mathfrak{H}] & [\mathfrak{J}] \\ & & [\mathfrak{K}] \end{bmatrix}
$$

$$
[[I]]_{33} = \frac{M}{15} c^2 \begin{bmatrix}
a^2 \begin{bmatrix} a^2 & 0 & 0 \\ & \frac{5}{9} b^2 & 0 \\ \text{sym} & & c^2 \end{bmatrix} & a^2 \begin{bmatrix} 0 & \frac{5}{9} b^2 & 0 \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix} & a^2 \begin{bmatrix} 0 & 0 & c^2 \\ & 0 & 0 \\ \text{sym} & & 0 \end{bmatrix} \\
& b^2 \begin{bmatrix} \frac{5}{9} a^2 & 0 & 0 \\ & b^2 & 0 \\ \text{sym} & & c^2 \end{bmatrix} & b^2 \begin{bmatrix} 0 & 0 & 0 \\ & 0 & c^2 \\ \text{sym} & & 0 \end{bmatrix} \\
\text{sym} & & c^2 \begin{bmatrix} a^2 & 0 & 0 \\ & b^2 & 0 \\ \text{sym} & & \frac{15}{7} c^2 \end{bmatrix}
\end{bmatrix}
$$

$$
= \begin{bmatrix} [\mathfrak{F}] & [\mathfrak{L}] & [\mathfrak{M}] \\ & [\mathfrak{K}] & [\mathfrak{N}] \\ \text{sym} & & [\mathfrak{O}] \end{bmatrix}
$$

and finally the non-diagonal clusters

$$
[[I]]_{12} = \begin{bmatrix} [\mathfrak{B}] & [\mathfrak{C}] & [\mathfrak{E}] \\ & [\mathfrak{G}] & [\mathfrak{I}] \\ \text{sym} & & [\mathfrak{L}] \end{bmatrix}
$$

$$
[[I]]_{13} = \begin{bmatrix} [\mathfrak{D}] & [\mathfrak{E}] & [\mathfrak{F}] \\ & [\mathfrak{I}] & [\mathfrak{L}] \\ \text{sym} & & [\mathfrak{M}] \end{bmatrix}
$$

$$
[[I]]_{23} = \begin{bmatrix} [\mathfrak{E}] & [\mathfrak{I}] & [\mathfrak{L}] \\ & [\mathfrak{J}] & [\mathfrak{H}] \\ \text{sym} & & [\mathfrak{N}] \end{bmatrix}
$$