NOAA Technical Report NOS 85 NGS 16

The Bruns Transformation and a Dual Setup of Geodetic Observational Equations

Rockville, Md.
April 1980

U.S. DEPARTMENT OF COMMERCE
National Oceanic and Atmospheric Administration
National Ocean Survey

NOAA Technical Publications

National Ocean Survey/National Geodetic Survey subseries

The National Geodetic Survey (NGS) of the National Ocean Survey (NOS), NOAA, establishes and maintains the basic National horizontal and vertical networks of geodetic control and provides governmentwide leadership in the improvement of geodetic surveying methods and instrumentation, coordinates operations to assure network development, and provides specifications and criteria for survey operations by Federal, State, and other agencies.

NGS engages in research and development for the improvement of knowledge of the figure of the Earth and its gravity field, and has the responsibility to procure geodetic data from all sources, process these data, and make them generally available to users through a central data base.

NOAA Technical Memorandums and some special NOAA publications are sold by the National Technical Information Service (NTIS) in paper copy and microfiche. Orders should be directed to NTIS, 5285 Port Royal Road, Springfield, VA 22161 (telephone: 703-557-4650). NTIS customer charge accounts are invited; some commercial charge accounts are accepted. When ordering, give the NTIS accession number (which begins with PB) shown in parentheses in the following citations.

Paper copies of NOAA Technical Reports, that are of general interest to the public, are sold by the Superintendent of Documents, U.S. Government Printing Office (GPO), Washington, DC 20402 (telephone: 202-783-3238). For prompt service, please furnish the GPO stock number with your order. If a citation does not carry this number, then the publication is not sold by GPO. All NOAA Technical Reports may be purchased from NTIS in hard copy and microform. Prices for the same publication may vary between the two Government sales agents. Although both are nonprofit, GPO relies on some Federal support whereas NTIS is self-sustained.

An excellent reference source for Government publications is the National Depository Library program, a network of about 1,300 designated libraries. Requests for borrowing Depository Library material may be made through your local library. A free listing of libraries currently in this system is available from the Library Division, U.S. Government Printing Office, 5236 Eisenhower Ave., Alexandria, VA 22304 (telephone: 703-557-9013).

NOAA geodetic publications

Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1974 reprinted annually, 12 pp (PB265442). National specifications and tables show the closures required and tolerances permitted for first-, second-, and third-order geodetic control surveys. (A single free copy can be obtained, upon request, from the National Geodetic Survey, OA/C18x2, NOS/NOAA, Rockville, MD 20852.)

Specifications To Support Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1975, reprinted annually, 30 pp (PB261037). This publication provides the rationale behind the original publication, "Classification, Standards of Accuracy, ..." cited above. (A single free copy can be obtained, upon request, from the National Geodetic Survey, OA/C18x2, NOS/NOAA, Rockville, MD 20852.)

Proceedings of the Second International Symposium on Problems Related to the Redefinition of North American Geodetic Networks. Sponsored by U.S. Department of Commerce; Department of Energy, Mines and Resources (Canada); and Danish Geodetic Institute; Arlington, Va., 1978, 658 pp. (GPO #003-017-0426-1). Fifty-four papers present the progress of the new adjustment of the North American Datum at midpoint, including reports by participating nations, software descriptions, and theoretical considerations.

NOAA Technical Memorandums, NOS/NGS subseries

NOS NGS-1 Use of climatological and meteorological data in the planning and execution of National Geodetic Survey field operations. Robert J. Leffler, December 1975, 30 pp (PB249677). Availability, pertinence, uses, and procedures for using climatological and meteorological data are discussed as applicable to NGS field operations.

NOS NGS-2 Final report on responses to geodetic data questionnaire. John F. Spencer, Jr., March 1976, 39 pp (PB254641). Responses (20%) to a geodetic data questionnaire, mailed to 36,000 U.S. land surveyors, are analyzed for projecting future geodetic data needs.

(Continued at end of publication)

NOAA Technical Report NOS 85 NGS 16

The Bruns Transformation and a Dual Setup of Geodetic Observational Equations

Erik W. Grafarend

National Geodetic Survey
Rockville, Md.
April 1980

U.S. DEPARTMENT OF COMMERCE
Philip M. Klutznick, Secretary

National Oceanic and Atmospheric Administration
Richard A. Frank, Administrator

National Ocean Survey
Herbert R. Lippold, Jr., Director

CONTENTS

Abstract 1

Introduction 2

1. Observables 3

2. The Bruns transformation 13

3. The dual setup of geodetic observational equations 32

Acknowledgment 46

Appendix A. Invariant representation of $||x - x'||^{-m}$ 47

Appendix B. Cartesian representation of the gravitational potential 50

Appendix C. Cartesian representation of first-order gradients of the gravitational potential 55

Appendix D. Cartesian representation of second-order gradients of the gravitational potential 58

Appendix E. Cartesian representation of the Euclidean norm of first-order gradients of the gravity potential 61

Bibliography 65

THE BRUNS TRANSFORMATION AND A DUAL SETUP OF GEODETIC OBSERVATIONAL EQUATIONS

Erik W. Grafarend[1]
National Geodetic Survey
National Ocean Survey, NOAA
Rockville, Md. 20852

ABSTRACT. The Bruns formula, which equates the disturbing gravity potential modulo the length of the normal gravity vector to the height anomaly, is generalized into three dimensions and into horizontal, equatorial, and inertial reference frames. It is applied to formulate the space-like geodetic boundary value problem in geometry and gravity space. The Bruns transform allows a dual setup of geodetic observational equations in a network of mass points, the finite element approximation of the space-like geodetic boundary value problem, in the following sense: The observational equations can be expressed rigorously either as a function of geometric coordinate corrections alone without any gravity dependent quantity, or alone as a function of the gravity disturbing potential and its gradients alone without any geometric coordinate correction. For operational purposes, estimable quantities from reference-free observables are studied in geometry, gravity, and vorticity spaces. They correspond to invariants with respect to a linear similarity transformation typified by positional angles and length ratios in various vector spaces. A Cartesian series representation of the gravity potential and its gradients is given--the Cartesian coordinate system is known to be singularity-free--and is used for a unified Cartesian setup of observational equations.

[1]Prepared during a 3-month period in 1978 when the author served as a Senior Scientist in Geodesy, National Research Council, National Academy of of Sciences, Washington, D.C., while on leave from the University FAF at Munich, Federal Republic of Germany.

> Science should be the friend of practice, but not its slave.
>
> C.F. Gauss

INTRODUCTION

Geodesy is conventionally divided into two branches: geometric and physical. This separation has resulted in various geodetic schools, or research groups, concentrating on one or the other aspect with little intercommunication. We would like to show that geodesy is actually a unity. The two branches are only the sides of a single coin. In detail, we will prove that geodetic observational equations can be uniquely set up in either the geometric or the physical mode. For instance, a distance observation can be expressed in terms of either the coordinates of the end points of a line or the gravity disturbing potential and its gradient at these points. The proof is based on the classic Bruns formula which expresses the height anomaly in terms of the gravity disturbing potential modulo the magnitude of the normal gravity vector. The Bruns formula will be generalized into three dimensions and into various reference frames: horizontal, equatorial, and inertial.

To make the Bruns formula operational, we have to inject observable quantities. Therefore, the first section is devoted to geodetic observables. There are two perspectives from which to look upon geodetic observables. If we do not introduce an a priori reference system into the vector space of geodesy, only positional angles and length ratios are observable. They are invariant with respect to a linear similarity transformation, characterized by degrees of freedom of type translation, rotation, and scale. Referring to adjustment procedures, positional angles and length ratios are estimable quantities. This concept is applied to both geometric and physical space. For instance, we construct positional angles and length ratios in gravity space from a network based on gravity vectors. The geometric quantities are a function of the length of the gravity vector and astronomical longitude and latitude at three points.

In the second section we will derive the generalized three-dimensional Bruns equation from observables that are one-point functions. These can be computed from observations once we have established a reference system for origin, orientation, and scale in any geodetic vector space. The first step will be a transformation of one-point observables into Cartesian coordinates of points on the approximate surface of the Earth, the telluroid. We will use isoparametric mappings for astronomical longitude and latitude, gravity potential, and first- and second-order gradients. The mappings are one-to-one if we use the isotropic- or zero-order approximation of the gravity field. Uniqueness is lost if we use another order of approximation. The second step is formulation of the transform of disturbances of gravity into Cartesian coordinate corrections.

The third section deals with a dual setup of geodetic observational equations of one- and three-point type, either in the geometric or in the gravitational space. They refer to different formulations of the Bruns transformation based on astronomical longitudes and latitudes and gravity potential (or gravity or gravity gradient).

The appendices are a Cartesian form of series representing the gravity potential and its first- and second-order gradients.

The report reflects current research in space-time geodesy, especially with respect to the geodetic initial-boundary value problem and its finite element approximation, the setup of geodetic observational equations in networks of mass points.

Section 1 is influenced by the concept of geodetic invariants introduced by Baarda (1973) and estimable quantities introduced by Bossler (1973) and Grafarend and Schaffrin (1974, 1976). The isoparametric mappings of section 2 which led to the formulation of the three dimensional Bruns transform have been partly studied by Bocchio (1976 a,b,c), Bruns (1876), Hirvonen (1960, 1961), Krarup (1969, 1973 a,b), Livieratos (1976, 1978), Marussi (1973, 1974 a,b), Moritz (1965, 1977), Niemeier (1972) and Grafarend (1972, 1975, 1978 a,b,c). The first setup of geodetic observational equations to be expressed rigorously in the gravimetric mode was by Sanso (1978 a,b) by making use of his adjoint potential. Here, we will reverse his argument exactly by employing the inverse Bruns transformation and expressing the geodetic observational potential rigorously in the geometric mode.

Geodesists have hesitated to accept the new three-dimensional mapping. Therefore we would like to make the following comments. For two-dimensional cartographic mappings the isoparametric mapping is well known, e.g. Chovitz (1952, 1954, 1956), O'Keefe (1953), Lane (1939, p. 189), Levi-Civita (1926, p. 220). Let us quote from O'Keefe (1953): "It is evident that the deformations produced by the isoparametric method are of the same order as those produced by other methods."

Another comment is on the definition of a geodetic network. Much research in geodesy has been performed in two-dimensional network analysis. Such networks are better termed mathematical networks because they do not take the gravity field into account. Here, a geodetic network consists of mass points; thus there is gravitational interaction which we cannot switch off.

1. OBSERVABLES

Having decided foundational questions, we next introduce related observations which make geodesy operational. A majority of geodesists believe that geodesy is Euclidean geometry referred to linear space with finite dimensions and Hilbertian geometry referred to linear space with infinite dimensions. What then are the basic observables?

In Euclidean geodesy position is given by vectors, for instance,

the position vector

the gravity vector

the vorticity vector

moving in space-time. Let us give an illustration of these vectors, as shown in figure 1 (p.70). Choose an origin of reference, e.g., the geocenter. The position vector extends from the reference origin to a mass point in space, e.g., the topocenter. At this point we draw the gravity vector, the rotation vector, or any other vector of reference. The corresponding vector spaces are called

the geometry space,

the gravity space, and

the vorticity space.

The set of all position vectors drawn from the reference origin is called the geometry space. Gravity space is constructed by a translation of the gravity vector along the position vector to the reference origin under Euclidean parallelism. In the same way the vorticity space or any other space of reference vectors is defined. Coordinates v^n of a vector $\underline{v}$ are provided after we select a frame of reference $\underline{e}_n$, e.g., the inertial frame, such that

$$\underline{v} = \sum_{n=o}^{N} v^n \underline{e}_n = v^o \underline{e}_o + v^1 \underline{e}_1 + \dots + v^N \underline{e}_N. \qquad 1(1)$$

(Notation: vectors in Euclidean space are denoted by capital letters, or underlined small letters.)

If the base vectors are orthonormal,

$$(\underline{e}_i, \underline{e}_j) = \delta_{ij} \qquad 1(2)$$

where δ_{ij} is the Kronecker symbol for an element of the unit matrix and (.,.) is the sign for the scalar product. Coordinates are recovered by

$$(\underline{v}, \underline{e}_n) = v^n. \qquad 1(3)$$

$$||\underline{v}||^2 = \sum_{n=o}^{N} (v^n)^2 \qquad 1(4)$$

(where $||\cdot||$ is the norm sign) is the relation of completeness.

It is assumed that in space-time geodesy the number of independent base vectors, which is identical to the dimension of the vector space, is (3,1).

In Hilbertian geodesy, position is given by vectors, for instance,

the position potential,

the gravity potential, and

the vorticity potential.

(In Hilbert space, potential is a vector.)

Figure 2 (p. 70) illustrates these vectors. Coordinates v^{nm} of a vector $\bar{v}$ (vectors in Hilbert space carry an overbar) are provided once we have select a frame of reference, e.g.,

$$\bar{e}_{nm} = \begin{cases} \sqrt{2n+1}\; r^{-n-1}\; P_n(\sin\phi) & \text{for } m=0 \\ \sqrt{2(2n+1)\,\frac{(n-m)!}{(n+m)!}}\; r^{-n-1}\; P_{nm}(\sin\phi)\cos m\lambda & \text{for } m>0 \\ \sqrt{2(2n+1)\,\frac{(n-|m|)!}{(n+|m|)!}}\; r^{-n-1}\; P_{nm}(\sin\phi)\sin m\lambda & \text{for } m<0 \end{cases} \qquad 1(5)$$

where P_n are Legendre and P_{nm} associated Legendre functions of the first kind, and λ, ϕ, r spherical coordinates, such that

$$\bar{v} = \sum_{n=o}^{\infty} \sum_{m=-n}^{+n} v^{nm}\, \bar{e}_{nm}. \qquad 1(6)$$

Here, $\bar{v}$ is a "harmonic" function satisfying $\Delta\bar{v}=0$, where Δ is the three-dimensional Laplace operator.

If the base vectors are orthonormal, as in our example,

$$(\bar{e}_{ij}, \bar{e}_{k\ell}) = \delta_{ik}\,\delta_{j\ell} \qquad 1(7)$$

e.g., the integral over the unit sphere divided by 4π. Coordinates are reproduced by

$$(\bar{v}, \bar{e}_{nm}) = v^{nm}. \qquad 1(8)$$

$$||\bar{v}|| = \sum_{n=o}^{\infty} \sum_{m=-n}^{+n} (v^{nm})^2 \qquad 1(9)$$

(where $||\bar{v}||^2$, the square of the norm, is the integral over the unit sphere of $(\bar{v},\bar{v})/4\pi$) is the relation of completeness.

The earlier question about basic observables can now be answered. Assume a network, e.g., a triangle, being constructed in a vector space. For the depiction of any vector by an arrow, as in figure 3 (p. 71), we require an origin, direction, and a length. To remove these artificial references for translation, rotation, and scale, we need quantities which are invariant with respect to changes of these parameters. In other words, we are looking for invariants under a similarity transformation

$$\underline{v} \rightarrow \underline{v}' = T + \lambda R\underline{v} \qquad 1(10)$$

where T is a translation vector, R an orthogonal matrix, and λ a scale factor. It is well known from analytical geometry that length ratios and angles are dual elements of the basis of invariants under the linear similarity transformation given above, e.g.,

$$\frac{||\underline{v}_2 - \underline{v}_1||}{||\underline{v}_3 - \underline{v}_1||} \qquad 1(11)$$

or

$$\frac{(\underline{v}_2 - \underline{v}_1, \underline{v}_3 - \underline{v}_1)}{||\underline{v}_2 - \underline{v}_1||\ ||\underline{v}_3 - \underline{v}_1||} \ . \qquad 1(12)$$

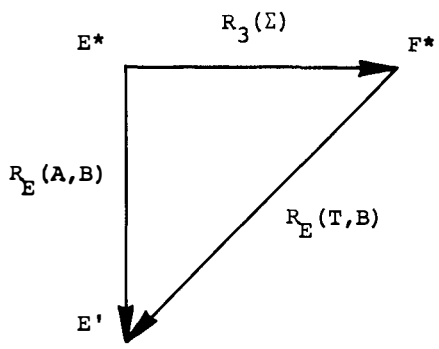
Finally, we will present three examples for basic observables in the geometry, gravity, and vorticity spaces.

EXAMPLE 1.1 (geometry space)

Let us introduce a triangle in the geometry space constructed from position vectors X_1, X_2, X_3 directed from the geocenter to three mass points in space.

At first the network is observed by a theodolite through horizontal directions and horizon distances (or zenith distances). Related reference frames are E', E*, and F*, defined as follows;

The orthonormal observational triad E' is based on the vector $E_{3'}$ directed from the station point at the topocenter to the target point. The base vector $E_{2'}$ is the normalized vector of the exterior product of the local, instantaneous gravity vector $-\Gamma$ and $E_{3'}$. $E_{1'}$ completes the orthonormal base. The orthonormal horizontal triad E* is based on the normalized local, instantaneous gravity vector E_{3*} at the topocenter. The base vector E_{2*} is the normalized vector of the exterior product of the local instantaneous rotation (vorticity) vector Ω and the local instantaneous gravity vector $-\Gamma$. E_{1*} completes the orthonormal base. The "carrousel" triad F* is related to the horizontal triad E* by $F^* = R_3(\Sigma)E^*$, where Σ is the horizontal orientation unknown such that F_{1*} is in the zero direction of the horizontal circle of the theodolite. To summarize, the frames are related as shown in the diagram



where $R_E(A,B) = R_2(\frac{\pi}{2} - B)R_3(A)$, and A the south azimuth, B the horizon distance, T the horizontal direction.

Now we can compute the positional angle

$$\cos \Psi_x = \frac{(X_2 - X_1,\ X_3 - X_1)}{||X_2 - X_1||\ ||X_3 - X_1||} = \left(E_{3'}^{12},\ E_{3'}^{13}\right). \qquad 1(13)$$

From the diagram we read

$$E' = R_E(A,B)E^* = R_E(A,B)R_3^T(\Sigma)F^* = R_E(T,B)F^* \qquad 1(14)$$

so that

$$E_{3'} = \cos T \cos B\ F_{1*} + \sin T \cos B\ F_{2*} + \sin B\ F_{3*} \qquad 1(15)$$

$$\cos \Psi_x = ||X_2 - X_1||^{-1}||X_3 - X_1||$$

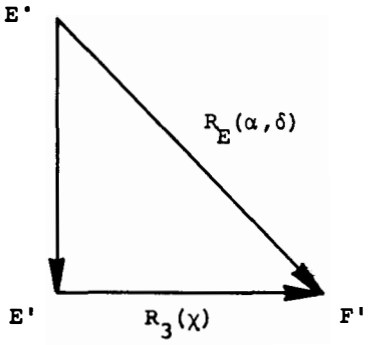
$$\left\{(X_{2*} - X_{1*})(X_{3*} - X_{1*}) + (Y_{2*} - Y_{1*})(Y_{3*} - Y_{1*}) + (Z_{2*} - Z_{1*})(Z_{3*} - Z_{1*})\right\}$$

$$= \cos T_{12} \cos B_{12} \cos T_{13} \cos B_{13} + \sin T_{12} \cos B_{12} \sin T_{13} \cos B_{13} + \sin B_{12} \sin B_{13}. \qquad 1(16)$$

Here, the positional angle is represented by horizontal directions and horizon distances at the three network points and is independent of the origin, orientation, and scale of the reference systems.

Next, the network is observed by a camera through right ascension and declination. Related reference frames are E', F', and E• which are defined as follows:

The orthonormal equatorial triad E• is based on the normalized local instantaneous rotation (vorticity) vector $E_{3\cdot}$ at the topocenter. The base vector $E_{2\cdot}$ is the normalized vector of the exterior product of the instantaneous ecliptic normal vector and the local instantaneous rotation (vorticity) vector. $E_{1\cdot}$ completes the orthonormal base. The "carrousel" triad F' is related to the observational triad E' by $F' = R_3(\chi)E'$, where χ is the observational orientation unknown. To summarize, the frames are related as shown in the diagram



where $R_E(\alpha,\delta) = R_2(\frac{\pi}{2} - \delta)\ R_3(\alpha)$, and α the right ascension, δ the declination.

From the diagram we see

$$F' = R_E(\alpha,\delta)E' \qquad 1(17)$$

so that

$$E_{3'} = \cos\alpha \cos\delta\ E_{1'} + \sin\alpha \cos\delta\ E_{2'} + \sin\delta\ E_{3'} \qquad 1(18)$$

$$\cos\Psi_X = ||X_2 - X_1||^{-1}\ ||X_3 - X_1||^{-1} \qquad 1(19)$$

$$\left\{(X_{2'} - X_{1'})(X_{3'} - X_{1'}) + (Y_{2'} - Y_{1'})(Y_{3'} - Y_{1'}) + (Z_{2'} - Z_{1'})(Z_{3'} - Z_{1'})\right\}$$

$$= \cos\alpha_{12} \cos\delta_{12} \cos\alpha_{13} \cos\delta_{13} + \sin\alpha_{12} \cos\delta_{12} \sin\alpha_{13} \cos\delta_{13} + \sin\delta_{12} \sin\delta_{13}$$

(where δ_{ij}, α_{ij} represent differences in δ, α between points i and j)

holds because $E_{3'} = F_{3'}$ by definition.

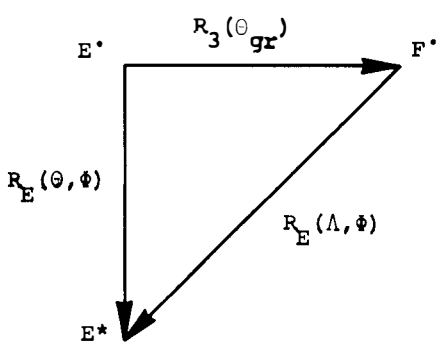
The positional angle above is represented by differences in right ascension and declination at the three network points and is independent of the origin, orientation, and scale of the reference systems.

EXAMPLE 1.2 (gravity space)

Let us introduce a triangle in the gravity space constructed from gravity vectors Γ_1, Γ_2, Γ_3.

The network is observed by an astronomic instrument and a gravimeter. Related reference frames are E*, E·, and F·, defined as follows

The frames E* and E· have been introduced in the first example. The "carrousel" triad F· is related to the equatorial triad E· by $F^{\cdot} = R_3\,(\Theta_{gr})\,E^{\cdot}$, where Θ_{gr} is the equatorial orientation angle, (also called Greenwich sidereal time), such that $F_{1\cdot}$ is in the Greenwich direction, the projection of the local instantaneous gravity vector at Greenwich onto the equatorial plane. The frames are related in the following manner:



where $R_E(\Theta,\Phi) = R_2(\frac{\pi}{2} - \Phi)R_3(\Theta)$, and Λ astronomic longitude, Φ astronomic latitude, Θ sidereal time angle.

We can again compute the positional angle

$$\cos \Psi_\Gamma = \frac{(\Gamma_2 - \Gamma_1,\ \Gamma_3 - \Gamma_1)}{||\Gamma_2 - \Gamma_1||\ ||\Gamma_3 - \Gamma_1||}. \qquad 1(20)$$

From the diagram,

$$E^* = R_E(\theta,\Phi)E^{\cdot} = R_E(\theta,\Phi) \quad R_3^T(\theta_{gr}) \; F^{\cdot} = R_E(\Lambda,\Phi)F^{\cdot} \qquad 1(21)$$

so that

$$E_{3*} = \cos\Lambda \cos\Phi \; F_{1\cdot} + \sin\Lambda \cos\Phi \; F_{2\cdot} + \sin\Phi \; F_{3\cdot} \qquad 1(22)$$

$$\Gamma = - ||\Gamma|| E_{3*} \qquad 1(23)$$

$$\cos\Psi_\Gamma = ||\Gamma_2 - \Gamma_1||^{-1} \; ||\Gamma_3 - \Gamma_1||^{-1} \qquad 1(24)$$

$$\Big\{[||\Gamma_2|| \cos\Lambda_2 \cos\Phi_2 - ||\Gamma_1|| \cos_1\Lambda \cos\Phi_1] \; [||\Gamma_3|| \cos\Lambda_3 \cos\Phi_3 - ||\Gamma_1|| \cos\Lambda_1 \cos\Phi_1] + [||\Gamma_2|| \sin\Lambda_2 \cos\Phi_2 - ||\Gamma_1|| \sin\Lambda_1 \cos\Phi_1] \; [||\Gamma_3|| \sin\Lambda_3 \cos\Phi_3 - ||\Gamma_1|| \sin\Lambda_1 \cos\Phi_1] + [||\Gamma_2|| \sin\Phi_2 - ||\Gamma_1|| \sin\Phi_1] \; [||\Gamma_3|| \sin\Phi_3 - ||\Gamma_1|| \sin\Phi_1]\Big\}.$$

Thus, the positional angle is represented by the magnitudes of the gravity vectors and astronomic longitude and latitude at three network points, and is independent of the origin, orientation, and scale of the reference systems.

EXAMPLE 1.3 (vorticity space)

Let us introduce a triangle in the rotation space constructed from rotation (vorticity) vectors Ω_1, Ω_2, Ω_3.

The network is observed with respect to a frame $f^{\cdot}$ uniformly rotating with rotational speed ω. $F^{\cdot}$, defined in the second example, and $f^{\cdot}$ are related by $F^{\cdot} = R_C(\gamma,\beta,\alpha)\; f^{\cdot}$, where $R_C(\gamma,\beta,\alpha) = R_1(\alpha)\; R_2(\beta)\; R_3(\gamma)$ are rotation matrices, and α,β,γ Cardan angles (Grafarend et al., 1979: p. 208).

The positional angle

$$\cos \Psi_\omega = \frac{(\Omega_2 - \Omega_1, \Omega_3 - \Omega_1)}{||\Omega_2 - \Omega_1|| \; ||\Omega_3 - \Omega_1||} \tag{1(25)}$$

can now be computed, taking into account

$$\begin{aligned} F_{3\cdot} = {} & (\sin \alpha \sin \gamma + \cos \alpha \sin \beta \cos \gamma) f_{1\cdot} \\ & - (\sin \alpha \cos \gamma + \cos \alpha \sin \beta \cos \gamma) f_{2\cdot} \\ & + \cos \alpha \cos \beta f_{3\cdot} \end{aligned} \tag{1(26)}$$

$$\Omega = ||\Omega|| F_{3\cdot} \tag{1(27)}$$

$$\begin{aligned} \cos \Psi_\omega = {} & ||\Omega_2 - \Omega_1||^{-1} \; ||\Omega_3 - \Omega_1||^{-1} \\ & \Big\{ \Big[||\Omega_2|| \; (\sin \alpha_2 \sin \gamma_2 + \cos \alpha_2 \sin \beta_2 \cos \gamma_2) \\ & - ||\Omega_1|| \; (\sin \alpha_1 \sin \gamma_1 + \cos \alpha_1 \sin \beta_1 \cos \gamma_1)] \\ & [||\Omega_3|| \; (\sin \alpha_3 \sin \gamma_3 + \cos \alpha_3 \sin \beta_3 \cos \gamma_3) \\ & - ||\Omega_1|| \; (\sin \alpha_1 \sin \gamma_1 + \cos \alpha_1 \sin \beta_1 \cos \gamma_1)] \\ & + [||\Omega_2|| \; (\sin \alpha_2 \cos \gamma_2 - \cos \alpha_2 \sin \beta_2 \cos \gamma_2) \\ & - ||\Omega_1|| \; (\sin \alpha_1 \cos \gamma_1 - \cos \alpha_1 \sin \beta_1 \cos \gamma_1)] \\ & [||\Omega_3|| \; (\sin \alpha_3 \cos \gamma_3 - \cos \alpha_3 \sin \beta_3 \cos \gamma_3) \\ & - ||\Omega_1|| \; (\sin \alpha_1 \cos \gamma_1 - \cos \alpha_1 \sin \beta_1 \cos \gamma_1)] \\ & + [||\Omega_2|| \cos \alpha_2 \cos \beta_2 - ||\Omega_1|| \cos \alpha_1 \cos \beta_1] \\ & [||\Omega_3|| \cos \alpha_3 \cos \beta_3 - ||\Omega_1|| \cos \alpha_1 \cos \beta_1] \Big\} \end{aligned} \tag{1(28)}$$

is represented as a function of the magnitude of the rotation (vorticity) vectors and the three Cardan angles at three network points and is independent of the origin, orientation, and scale of the reference systems.

2. THE BRUNS TRANSFORMATION

Operational geodesy uses observables as input data and coordinates of the position vector as output data. This input-output relation is called the Bruns transformation, originally presented in its linear form by Bruns (1878) when referring to the horizontal frame. The Bruns transformation classically yields the height anomaly from the disturbing potential divided by normal gravity. Thus it transforms the "observable" disturbing potential into the vertical coordinate called "height." Now, we will present a three-dimensional generalization of the Bruns transformation which can be used in both terrestrial and satellite geodesy.

The idea of the Bruns transformation is the following: Let a vector field $\bar{V}$ be observed, for instance, at a point P on the Earth's surface. Decompose the vector field into a normal part, whose structure is known (which approximates the real vector field), and into a disturbing part:

$$\bar{V}_P = \bar{v}_p + \delta\bar{v}_P. \qquad 2(1)$$

The normal part $\bar{v}_P$ at P can be linearized by a Taylor series with origin at a point p of the telluroid:

$$\bar{v}_P = \bar{v}_p + (\text{grad } \bar{v}_P)_p \ (P-\underline{p}) + o_2 \qquad 2(2)$$

where o_2 indicates second- and higher-order terms. If we know the approximate position vector $\bar{p}$, we can determine the "displacement vector" $P-\underline{p}$ from

$$\bar{V}_P - \bar{v}_p = (\text{grad } \bar{v}_P)_p \ (P-\underline{p}) + \delta\bar{v}_P. \qquad 2(3)$$

Figure 4 (p.71) illustrates the vector field in geometric space. Let us call the two-point functions, $\bar{V}_P - \bar{v}_p = \Delta\bar{v}$ and $P-\underline{p} = \Delta\underline{x}$, anomalies of the vector field and the position vector, respectively, so that

$$\Delta\bar{v} = (\text{grad } \bar{v}_P)_p \ \Delta\underline{x} + \delta\bar{v}_P. \qquad 2(4)$$

We can choose $\Delta\bar{v} = 0$ which we refer to as the isoparametric mapping of $\delta\bar{v}_p \rightarrow \Delta\underline{x}$

$$\Delta\underline{x} = B\delta\bar{v}, \quad \delta\bar{v} = C\Delta\underline{x}, \tag{2(5)}$$

where

$$B = C^{-1} = -\,(\text{grad}\ \bar{v}_p)_p^{-1}, \tag{2(6)}$$

the Brun's matrix, which is the inverse of the gradient of the vector field at point p. Here we have assumed that grad $\bar{v}P$ is a regular matrix excluding rank deficiencies with respect to injectivity. In practice, singularities appear and have to be treated separately.

Thus far the Bruns transformation 2(5) of vector field disturbances into position vector anomalies is coordinate-free. Its form with respect to geodetic reference frames is the following:

$$\Delta\underline{x}^{\cdot} = B^{\cdot}\delta\bar{v} \tag{2(7)}$$

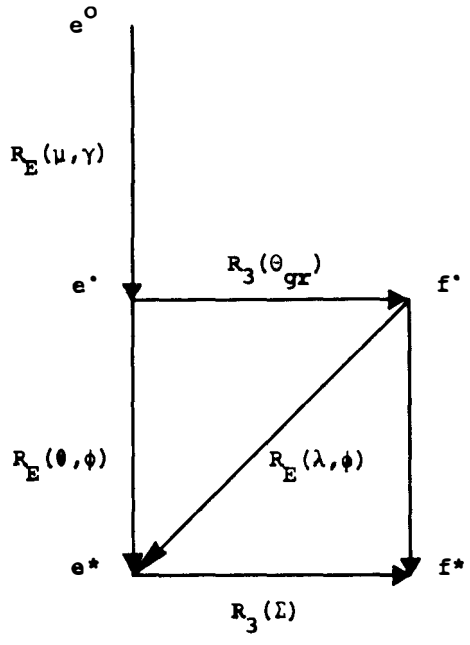
$$\Delta\underline{x}^{*} = B^{*}\delta\bar{v} \tag{2(8)}$$

$$\Delta\underline{x}^{o} = B^{o}\delta\bar{v} \tag{2(9)}$$

$$B^{*} = R_E(\lambda,\phi)B^{\cdot} \tag{2(10)}$$

$$B^{o} = R_E^T(\mu,\nu)\ R_E^T(\theta_{gr})B^{\cdot}\ . \tag{2(11)}$$

$\Delta\underline{x}^{\cdot} = [\Delta x^{\cdot}, \Delta y^{\cdot}, \Delta z^{\cdot}]^T$ displays the coordinates of the "displacement vector" in the Earth-fixed equatorial triad $f^{\cdot}$, $\Delta\underline{x}^{*} = [\Delta x^{*}, \Delta y^{*}, \Delta z^{*}]^T$, and the corresponding coordinates in the horizon triad e^{*}, $\Delta\underline{x}^{o} = [\Delta x^{o}, \Delta y^{o}, \Delta z^{o}]^T$ in the "fixed" or inertial (ecliptic) triad e^{o}, where μ,ν are Eulerian angles, as shown in the following diagram:



Instead we could represent the "displacement vector" in the "network" frames (written by capital letters). In the theory of mappings the two frames are called Eulerian and Lagrangian; thus we have here the Eulerian description.

Before we show examples of the thus far abstract Bruns transformation, we first note another remarkable property.

In many applications, the vector field of observables is "conservative," i.e.,

$$\text{div}\ \delta\bar{v} = 0, \ \text{rot}\ \delta\bar{v} = 0 \qquad 2(12)$$

at least insofar as we are outside of the masses. A consequence of the Bruns transformation 2(5) is then

$$\text{div}\ \Delta\underline{x} = 0, \ \text{rot}\ \Delta\underline{x} = 0; \qquad 2(13)$$

thus $\delta\bar{v}$ and $\Delta\underline{x}$ can be expressed as the gradient of a scalar potential. If we introduce

$$\delta\bar{v} = \text{grad } \delta w, \tag{2(14)}$$

$$\Delta\underline{x} = \text{grad } \delta x = B \text{ grad } \delta w \tag{2(15)}$$

follows. δx will be called the adjoint potential. In the first instance, it might be surprising that the "displacement vector" which leads to the position vector is a "harmonic" function, div $\Delta\underline{x}$ = div grad $\delta x = 0$, but the result is quite "natural."

Another basic assumption we have made is that we know the approximate position vector $\underline{p}$, but from where? If we have settled a convention about origin, orientation, and scale of a geodetic network to, for example, the geocenter, directions to extragalactic objects and a unit length in geometry space, how can we find the approximate position vector $\underline{p}$ in a geodetic reference system? The factor of uncertainty is introduced by the fact that nearly all geodetic observables depend on the gravity field whose coordinates in Hilbert space (e.g., coefficients in a spherical harmonic representation of the gravitational field) are unknown. Fortunately, there are geodetic observables that are gravity free, like positional angles and distance ratios. Only because of this can geodesy be made operational: coordinates in Euclidean and Hilbert spaces can be determined. This general statement will be verified in example 2.1.

EXAMPLE 2.1 (longitude, latitude, potential)

Let us introduce the isoparametric mapping

$$\Lambda_P = \lambda_p, \ \Phi_P = \phi_p, \ W_P = w_p \tag{2(16)}$$

where Λ and Φ are astronomic longitude and latitude, W the scalar gravity potential, λ and ϕ geodetic longitude and latitude, and w the scalar normal gravity potential (better known in geodesy by the letter U). Longitude and latitude are spherical coordinates in gravity space defined by

$$\Lambda = \text{arc tan } \Gamma_y/\Gamma_x \tag{2(17)}$$

$$\lambda = \text{arc tan } \gamma_y/\gamma_x \tag{2(18)}$$

$$\Phi = \text{arc tan } \Gamma_z/\sqrt{\Gamma_x^{\ 2} + \Gamma_y^{\ 2}} \tag{2(19)}$$

and

$$\phi = \arctan \gamma_z / \sqrt{\gamma_x^2 + \gamma_y^2} \qquad 2(20)$$

where $(\Gamma_x, \Gamma_y, \Gamma_z)$, $(\gamma_x, \gamma_y, \gamma_z)$ are Cartesian coordinates of the gravity vector Γ, $\underline{\gamma}$, respectively, in the "Eulerian" frame $\underline{f}^{\cdot}$. With respect to a chosen reference system, (Λ, Φ, W) are "observable," whereas (λ, ϕ, w) are "computable," as in the representation of the potential given in the first section.

A zero-order approximation of the actual gravity potential is

$$w = - gm \; ||\underline{x}||^{-1} \qquad 2(21)$$

where gm is the product of the gravitational constant and the mass of the model terrestrial body, $||\underline{x}|| = \sqrt{x^2 + y^2 + z^2}$, the length of the vector $\underline{x}$.

$$\partial w / \partial x = gm \; ||\underline{x}||^{-3} \; x \qquad 2(22)$$

(and similarly for y and z).

$$\Lambda = \arctan y/x \qquad 2(23)$$

$$\Phi = \arctan z / \sqrt{x^2 + y^2} \qquad 2(24)$$

and

$$W = gm / \sqrt{x^2 + y^2 + z^2} \qquad 2(25)$$

are the corresponding zero-order mapping equations. They can be inverted into

$$x = \frac{gm}{W} \cos \Lambda \cos \Phi \qquad 2(26)$$

$$y = \frac{gm}{W} \sin \Lambda \cos \Phi \qquad 2(27)$$

$$z = \frac{gm}{W} \sin \Phi \qquad 2(28)$$

(excluding, of course, $\phi = \pm \pi/2$). If we know (Λ, Φ, W) and (gm) from observed quantities, Cartesian coordinates of the approximate position vector can be computed. But how can we call the quantities (Λ, Φ, W, gm) observable?

Astronomical longitude and latitude are quantities derived from observations related to geodetic astronomy. Fundamental catalogs and a variety of reductions (precession, nutation, polar motion, aberration, parallax, etc.) are involved. The setup of an observational equation in geodetic astronomy is not routine and assumes, strictly speaking, approximate a priori information about the position of an observer in geometry space. From gravimetric leveling we obtain only potential differences. In order to be able to derive absolute potential, the reference system should contain sufficient information in its definition. In addition, if we introduce length observations and we extend the isoparametric mapping to the isometric case in geometry space by

$$\overline{P_1P_2} = \overline{p_1p_2}\,, \qquad 2(29)$$

for example, by imposing equal length of an observational line between two points on the Earth's surface and the corresponding points on the telluroid, we will be able to determine a value for gm. More details will be given in section 3.

The next step is a computation of the Bruns matrix.

$$\underline{v} = \begin{bmatrix} \lambda \\ \phi \\ w \end{bmatrix}, \quad \delta\underline{v} = \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w \end{bmatrix} \qquad 2(30)$$

$$C^{\cdot} = - \begin{bmatrix} \partial\lambda/\partial x & \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{bmatrix} \qquad 2(31)$$

$$= \begin{bmatrix} -\dfrac{y}{x^2+y^2} & +\dfrac{x}{x^2+y^2} & 0 \\ -\dfrac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} & -\dfrac{yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} & +\dfrac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} \\ -\dfrac{gmx}{(x^2+y^2+z^2)^{3/2}} & -\dfrac{gmy}{(x^2+y^2+z^2)^{3/2}} & -\dfrac{gmz}{(x^2+y^2+z^2)^{3/2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{W}{gm}\dfrac{\sin\Lambda}{\cos\Phi} & +\dfrac{W}{gm}\dfrac{\cos\Lambda}{\cos\Phi} & 0 \\ -\dfrac{W}{gm}\cos\Lambda\sin\Phi & -\dfrac{W}{gm}\sin\Lambda\sin\Phi & \dfrac{W}{gm}\cos\Phi \\ -\dfrac{W^2}{(gm)^2}\cos\Lambda\cos\Phi & -\dfrac{W^2}{(gm)^2}\sin\Lambda\cos\Phi & \dfrac{W^2}{(gm)^2}\sin\Phi \end{bmatrix}$$

$$B^{\cdot} = C^{\cdot -1}$$

$$A^* = C^{\cdot}\, R_E^T(\lambda,\phi)$$

$$B^* = C^{*^{-1}}$$

$$A^o = C^o \; R_E(\theta_{gr}) \; R_E(\mu,\gamma)$$

$$B^o = C^{o^{-1}}$$

where the dot, asterisk, and circle denote equatorial, horizontal, and ecliptic coordinates, respectively.

A zero-order approximation of the vector $\Delta \underline{x}^*$ is

$$\Delta x^* = 0 \qquad 2(32)$$

$$\Delta y^* = 0 \qquad 2(33)$$

$$\Delta z^* = \frac{gm}{W^2} \delta w \qquad 2(34)$$

which corresponds to the original Bruns formula, because within the zero-order approximation

$$\gamma = \frac{gm}{||\underline{x}||^2} = \frac{W^2}{gm} \quad . \qquad 2(35)$$

Finally, figure 5 (p. 71) illustrates the isoparametric mapping in the curvi-linear gravity and Cartesian geometry space.

EXAMPLE 2.2 (longitude, latitude, gravity)

Let us introduce the isoparametric mapping

$$\Lambda_P = \lambda_p, \quad \Phi_P = \phi_p, \quad \Gamma_P = \gamma_p \qquad 2(36)$$

where Λ and Φ are astronomic longitude and latitude, Γ the length of the gravity vector, λ and ϕ geodetic longitude and latitude, and γ the length of the normal gravity vector. We refer also to (Λ, Φ, Γ) as the spherical coordinates of the gravity vector. Alternatively, a Cartesian representation of 2(36) is the vector identity

$$\underline{\Gamma}_P = \underline{\gamma}_p \qquad 2(37)$$

or

$$\Gamma_x = \gamma_x \qquad 2(38)$$

$$\Gamma_y = \gamma_y \qquad 2(39)$$

$$\Gamma_z = \gamma_z \qquad 2(40)$$

in an "Eulerian" frame f'. Because

$$\Gamma_x = \Gamma \cos \Lambda \cos \Phi \qquad 2(41)$$

$$\Gamma_y = \Gamma \sin \Lambda \cos \Phi \qquad 2(42)$$

$$\Gamma_z = \Gamma \sin \Phi \qquad 2(43)$$

the spherical and the Cartesian mappings are equivalent (excluding again $\phi = \pm \pi/2$).

Corresponding to the first example, astronomical longitude and latitude are observed by astronomical instruments, and the length of the gravity vector by gravimeters. In addition, the scale of gravity space has to be included in the reference system.

A zero-order approximation of the actual gravity potential is

$$w = - gm \, ||\underline{x}||^{-1} \tag{2(44)}$$

$$\partial w/\partial x = - gm \, ||\underline{x}||^{-3} x = \gamma \tag{2(45)}$$

and similarly for y and z.

$$\Lambda = \text{arc tan } y/x \tag{2(46)}$$

$$\Phi = \text{arc tan } z/\sqrt{x^2+y^2} \tag{2(47)}$$

$$\Gamma = gm \, ||\underline{x}||^{-2} \tag{2(48)}$$

are the global mapping equations. They can be inverted into

$$x = \sqrt{\frac{gm}{\Gamma}} \cos \Lambda \cos \Phi \tag{2(49)}$$

$$y = \sqrt{\frac{gm}{\Gamma}} \sin \Lambda \cos \Phi \tag{2(50)}$$

$$z = \sqrt{\frac{gm}{\Gamma}} \sin \Phi \tag{2(51)}$$

(excluding $\phi = \pm \pi/2$). Compare 2(26), 2(27), 2(28) to 2(49), 2(50), 2(51) to see that only the "radial" component has changed from gm W^{-1} to $\sqrt{gm \, \Gamma^{-1}}$.

A "Cartesian" proof of 2(49), 2(50), 2(51) follows. Starting with global mapping equations

$$\Gamma_x = \gamma_x = - gm \, ||\underline{x}||^{-3} \, x \qquad 2(52)$$

$$\Gamma_y = \gamma_y = - gm \, ||\underline{x}||^{-3} \, y \qquad 2(53)$$

and

$$\Gamma_z = \gamma_z = - gm \, ||\underline{x}||^{-3} \, z \; . \qquad 2(54)$$

We write these in the general form:

$$A = \frac{a}{(x^2+y^2+z^2)^{3/2}} \, x \qquad 2(55)$$

$$B = \frac{a}{(x^2+y^2+z^2)^{3/2}} \, y \qquad 2(56)$$

$$C = \frac{a}{(x^2+y^2+z^2)^{3/2}} \, z \quad . \qquad 2(57)$$

Insert 2(57) into 2(55) and 2(56) to derive

$$A = \frac{C}{z} \, x \qquad 2(58)$$

$$B = \frac{C}{z} \, y \qquad 2(59)$$

or

$$x = AC^{-1} \, z \qquad 2(60)$$

$$y = BC^{-1} \, z \quad . \qquad 2(61)$$

Writing 2(57) as

$$(x^2+y^2+z^2)^{3/2} = C^{-1}\ az \qquad 2(62)$$

leads to

$$(A^2C^{-2} + B^2C^{-2} + 1)^{3/2}\ z^3 = C^{-1}\ az \qquad 2(63)$$

$$(A^2 + B^2 + C^2)^{3/2}\ C^{-3}\ z^2 = C^{-1}\ a \qquad 2(64)$$

$$z^2 = a\ C^2\ (A^2 + B^2 + C^2)^{-3/2} \qquad 2(65)$$

$$x = \Gamma_x\ \Gamma_z^{-1}\ z \qquad = \sqrt{\frac{gm}{\Gamma}} \cos \Lambda \cos \Phi \qquad 2(66)$$

$$y = \Gamma_y\ \Gamma_z^{-1}\ z \qquad = \sqrt{\frac{gm}{\Gamma}} \sin \Lambda \cos \Phi \qquad 2(67)$$

$$z = (\pm) \sqrt{gm}\ \Gamma_z \left(\Gamma_x^{\ 2} + \Gamma_y^{\ 2} + \Gamma_z^{\ 2}\right)^{-3/4} = \sqrt{\frac{gm}{\Gamma}} \sin \Phi \quad . \qquad 2(68)$$

A first-order approximation of the actual gravity potential is

$$w = -\ gm\ ||\underline{x}||^{-1} - \omega^2\ (x^2 + y^2) \qquad 2(69)$$

where ω is the length of the rotation (vorticity) vector.

$$\partial w/\partial x = \gamma_x = (+\ gm\ ||\underline{x}||^{-3} - 2\omega^2)\ x \qquad 2(70)$$

$$\partial w/\partial y = \gamma_y = (+ gm \, ||\underline{x}||^{-3} - 2\omega^2) \, y \qquad 2(71)$$

$$\partial w/\partial z = \gamma_z = + gm \, ||\underline{x}||^{-3} \, z \quad . \qquad 2(72)$$

Equations 2(70), 2(71), 2(72) can be written in the general form:

$$A = \left[\frac{a}{(x^2+y^2+z^2)^{3/2}} + b \right] x \qquad 2(73)$$

$$B = \left[\frac{a}{(x^2+y^2+z^2)^{3/2}} + b \right] y \qquad 2(74)$$

$$C = \frac{a}{(x^2+y^2+z^2)^{3/2}} \, z \quad . \qquad 2(75)$$

Insert 2(75) into 2(73) and 2(74) to obtain

$$A = (\frac{C}{z} + b) \, x \qquad 2(76)$$

$$B = (\frac{C}{z} + b) \, y \qquad 2(77)$$

or

$$x = Az \, (B + bz)^{-1} \qquad 2(78)$$

$$y = Bz \, (C + bz)^{-1} \qquad 2(79)$$

which, together with 2(57), is written as

$$(x^2 + y^2 + z^2)^3 = C^{-2}\, a^2\, z^2 \quad , \tag{2(80)}$$

leads to

$$z^4[(A^2 + B^2) + (C + bz)^2]^3 = C^{-2}\, a^2\, (C + bz)^6 \quad . \tag{2(81)}$$

This is an equation of the tenth order, i.e., of the form

$$z^{10} + \alpha z^9 + \ldots + \beta = 0 \quad . \tag{2(82)}$$

Equation 2(82) gives a set of solutions for z, then 2(78) for x, and 2(79) for y. Thus we have inverted 2(38), 2(39), 2(40) in gravity space into equations in geometry space. Of course, the inversion is not single-valued, but the solution space can be easily obtained.

The next setup will be a computation of the Bruns matrix.

$$\underline{\gamma} = \begin{bmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{bmatrix} , \quad \delta\underline{\gamma} = \begin{bmatrix} \delta\gamma_x \\ \delta\gamma_y \\ \delta\gamma_z \end{bmatrix} \tag{2(83)}$$

$$C^{\bullet} = -\partial\underline{\gamma}/\partial\underline{x} \tag{2(84)}$$

$$C^{\bullet} = 3\ gm\ ||\underline{x}||^{-5} \begin{bmatrix} x - \frac{1}{3}||\underline{x}||^2 + 2\omega^2 & xy & xz \\ xy & y^2 - \frac{1}{3}||\underline{x}||^2 + 2\omega^2 & yz \\ xz & yz & z^2 - \frac{1}{3}||\underline{x}||^2 \end{bmatrix} \tag{2(85)}$$

to a first-order approximation.

Because the gravity vector field to the first order is conservative, div $\delta \underline{v} = 0$, rot $\delta \underline{v} = 0$ leads via the Bruns transformation to div $\Delta \underline{x} = 0$, rot $\Delta \underline{x} = 0$. Thus if $\delta \underline{v}$ is taken from the space of spherical harmonics, so is $\Delta \underline{x}$.

Figure 6 (p. 72) illustrates the isoparametric mapping in the Cartesian gravity and geometric space.

EXAMPLE 2.3 (longitude, latitude, gravity gradient)

Let us introduce the isoparametric mapping

$$\Lambda_P = \lambda_p, \ \Phi_P = \phi_p, \ W_{ij}{}^P = w_{ij}{}^p \qquad 2(86)$$

where Λ and Φ are astronomic longitude and latitude, W_{ij} second-order gradients of the actual gravity potential (i,j ranging over x,y,z), λ and ϕ geodetic longitude and latitude, and w_{ij} second-order gradients of the normal gravity potential in the horizontal triad. Specific gravity gradients W_{xz}, W_{yz}, W_{xy} and $W_\Delta = W_{yy} - W_{xx}$ are assumed to be measured by a torsion balance, or any W_{ij} by a Gradiometer.

The first problem is to find a representation of model gravity gradients in the equatorial triad. Because of the transformation $f^\cdot \to e^* = R_E\ (\lambda,\phi)\ f^\cdot$, the first-order gradient tensor (grad $\underline{\gamma}$) can be determined by

$$(\text{grad } \underline{\gamma})^* = R_E(\lambda,\phi)\ (\text{grad } \underline{\gamma})^\cdot\ R_E^T\ (\lambda,\phi)\,. \qquad 2(87)$$

For instance ,

$$\begin{aligned} w_{x^*z^*} = &- 3\ gm\ ||\underline{x}||^{-5}\ (x^2 - z^2)\ \cos\lambda\ \sin\phi\ \cos\phi \qquad 2(88) \\ &+ 3\ gm\ ||\underline{x}||^{-5}\ xz\ \cos\lambda\ \cos 2\phi \\ &+ 3\ gm\ ||\underline{x}||^{-5}\ xy\ \sin\lambda\ \cos\phi \\ &+ 3\ gm\ ||\underline{x}||^{-5}\ yz\ \sin\lambda\ \sin\phi \end{aligned}$$

$$w_{y^*z^*} = + 3\ gm\ ||\underline{x}||^{-5}\ (x^2 - z^2)\ \sin\lambda\ \sin\phi\ \cos\phi$$
$$- 3\ gm\ ||\underline{x}||^{-5}\ xz\ \sin\phi\ \cos 2\phi$$
$$+ 3\ gm\ ||\underline{x}||^{-5}\ xy\ \cos\lambda\ \cos\phi$$
$$+ 3\ gm\ ||\underline{x}||^{-5}\ yz\ \cos\lambda\ \sin\phi \qquad 2(89)$$

$$w_{x^*y^*} = + 3\ gm\ ||\underline{x}||^{-5}\ x^2\ \sin\lambda\ \cos\lambda\ \sin^2\phi$$
$$- 3\ gm\ ||\underline{x}||^{-5}\ y^2\ \sin\lambda\ \cos\lambda$$
$$+ 3\ gm\ ||\underline{x}||^{-5}\ z^2\ \sin\lambda\ \cos\lambda\ \cos^2\phi$$
$$+ 3\ gm\ ||\underline{x}||^{-5}\ xy\ \cos 2\lambda\ \sin\phi$$
$$- 3\ gm\ ||\underline{x}||^{-5}\ xz\ \sin 2\lambda\ \sin\phi\ \cos\phi$$
$$- 3\ gm\ ||\underline{x}||^{-5}\ yz\ \cos 2\lambda\ \cos\phi\ . \qquad 2(90)$$

If we choose the zero-order approximation of the normal gravity potential

$$w = + gm\ ||\underline{x}||^{-1} \qquad 2(91)$$

the gravity gradients are given by

$$w_{ij} = - 3\ gm\ ||\underline{x}||^{-5}\ \left[(\underline{e}_i^T\underline{x})(\underline{e}_j^T\underline{x}) - \frac{1}{3}||\underline{x}||^2\ \delta_{ij}\right] \quad . \qquad 2(92)$$

Examples of 2(92) are

$$w_{x^\cdot x^\cdot} = - 3\ gm\ ||\underline{x}||^{-5}\ \left[\frac{2}{3}\ x^2 - \frac{1}{3}\ y^2 - \frac{1}{3}\ z^2\right] \qquad 2(93)$$

$$w_{x^\cdot y^\cdot} = - 3\ gm\ ||\underline{x}||^{-5}\ xy \quad . \qquad 2(94)$$

If we are interested in the zero-order approximation 2(91), the isoparametric mapping equations 2(86) can be summed up to be

$$\Lambda = \text{arc tan } y/x \tag{2(95)}$$

$$\Phi = \text{arc tan } z/\sqrt{x^2+y^2} \tag{2(96)}$$

$$W_{ij} = w_{ij}\ (x,\ y,\ z)\ . \tag{2(97)}$$

The general solution can be represented by

$$x = ||\underline{x}||\ \cos\Lambda\ \cos\Phi \tag{2(98)}$$

$$y = ||\underline{x}||\ \sin\Lambda\ \cos\Phi \tag{2(99)}$$

$$z = ||\underline{x}||\ \sin\Phi\ . \tag{2(100)}$$

Scale is taken from

$$W_{ij} = gm\ ||\underline{x}||^{-3}\ f_{ij}\ (\Lambda,\Phi) \tag{2(101)}$$

$$||\underline{x}|| = \sqrt[3]{f_{ij}\ (\Lambda,\Phi)} \qquad \sqrt[3]{\frac{gm}{W_{ij}}} \qquad \text{(no summation)} \tag{2(102)}$$

where f_{ij} (Λ,ϕ) is a specific expression, an example of which, for i=x and j=z, is shown below. Substituting equations 2(98) to 2(100) into 2(88) yields

$$\begin{aligned} w_{x^*z^*} &= +\ 3\ gm\ ||\underline{x}||^{-3}\ \{(\sin^2\Phi - \cos^2\Lambda\ \cos^2\Phi)\ \cos\Lambda\ \sin\Phi\ \cos\Phi \\ &\quad + \cos^2\Lambda\ \sin\Phi\ \cos\Phi\ \cos 2\Phi + \sin^2\Lambda\ \cos\Lambda\ \cos^3\Phi + \sin^2\Lambda\ \sin^2\Phi\ \cos\Phi\} \\ &= gm\ ||\underline{x}||^{-3}\ f_{xz}(\Lambda,\Phi)\,. \end{aligned} \tag{2(103)}$$

Compare 2(26) - 2(28), 2(49) - 2(51) and 2(98) - 2(103) in order to see that only the "radial" component has changed in the following way:

$$\frac{gm}{W} \tag{2(104)}$$

$$\sqrt{\frac{gm}{\Gamma}} \tag{2(105)}$$

$$\sqrt[3]{\frac{gm}{W_{ij}}}\ f_{ij}\ (\Lambda,\Phi) \qquad \text{(no summation).} \tag{2(106)}$$

Thus any coordinate of the Cartesian tensor of gravity gradients is as easily chosen as another.

In addition to the isoparametric mapping of 2(86), I tried one with only the gravity gradients mapped isoparametrically, but the mapping equations turned out extremely nonlinear and I have been unable to invert them.

The next step is the computation of the Bruns matrix.

$$v = \begin{bmatrix} \lambda \\ \phi \\ w_{ij} \end{bmatrix}, \qquad \delta v = \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w_{ij} \end{bmatrix} \tag{2(107)}$$

$$C^{\cdot} = - \begin{bmatrix} \partial\lambda/\partial x & \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \\ \partial w_{ij}/\partial x & \partial w_{ij}/\partial y & \partial w_{ij}/\partial z \end{bmatrix}, \tag{2(108)}$$

The first two rows of the matrix $C^{\cdot}$ were computed within 2(31); in addition

$$\partial w_{ij}/\partial x,\ \partial w_{ij}/\partial y,\ \partial w_{ij}/\partial z \tag{2(109)}$$

have to be computed. Using 2(92) we arrive at

$$\partial w_{ij}/\partial \underline{x} = \qquad 2(110)$$

$$-3\ gm\ ||\underline{x}||^{-7} \left\{-5\ \underline{e}_i^T\underline{x}\ \underline{e}_j^T\underline{x}\ \underline{x}^T + ||\underline{x}||^2 \left(\delta_{ij}\underline{x}^T + \underline{e}_i^T\underline{x}\ \underline{e}_j^T + \underline{e}_j^T\underline{x}\ \underline{e}_i^T\right)\right\}$$

$$B^{\cdot} = C^{\cdot -1} \quad .$$

Finally, figure 7 (p. 72) illustrates the isoparametric mapping in the generalized gravity and geometric space.

Let us summarize what the examples tell us. If we refer to the isotropic (zero-order) approximation of the normal gravity field

$$w = -\ gm\ ||\underline{x}||^{-1}$$

we can represent zero-order coordinates of telluroid points by

$$x = ||\underline{x}||\ \cos\Lambda\ \cos\Phi$$

$$y = ||\underline{x}||\ \sin\Lambda\ \cos\Phi$$

$$z = ||\underline{x}||\ \sin\Phi$$

where scale is taken from a quantity referring to the gravity field, like $||\underline{x}||$, as given by 2(104), 2(105), 2(106). For a higher order normal gravity field the (x, y, z) representation is more complicated as can be inferred from 2(78), 2(79) and 2(82). In addition, the examples emphasize the nearly arbitrary choice of the isoparametric mapping $p \rightarrow P$. As we will see in the next chapter, the isoparametric mapping

$$\Lambda_P = \lambda_p,\ \Phi_P = \phi_p,\ W_P = w_p$$

leads to the Stokes approximation of the geodetic boundary value problem and its finite element form.

3. THE DUAL SETUP OF GEODETIC OBSERVATIONAL EQUATIONS

Once we have decided upon the reference system in either the geometry, gravity, or vorticity space, we are able to set up geodetic observational equations. In general, these depend on coordinates in these spaces. Let us assume for a moment we know the approximate coordinates such that we can linearize observational equations. The quantities "observed minus computed" $\underline{Y}_P - \underline{y}_p$ can be represented by the gradients with respect to these coordinates, such as

$$\underline{Y}_P - \underline{y}_p = (\mathrm{grad}_x\ \underline{y}_P)_p\ \Delta\underline{x} + \delta\underline{y}_p \qquad 3(1)$$

where δy is the disturbance vector. There are geodetic observational equations which depend on coordinates only of the geometry space, but, in general, they are a function of gravity space coordinates of Hilbert type, e.g., spherical harmonic coefficients.

To present the idea of dual setup of geodetic observational equations in a simple way, we will start with a priori parameters which describe the normal gravity field, e.g., (gm).

If we use the dual Bruns transformation

$$\Delta\underline{x} = B\ \delta\underline{v} \qquad 3(2)$$

$$\delta\underline{v} = C\ \Delta\underline{x} \qquad 3(3)$$

which expresses coordinate corrections in the geometry space in terms of disturbances in the gravity space (with $BC = I$, $\det B \neq 0$, $\det C \neq 0$) to replace geometric coordinate corrections by gravimetric coordinate disturbances and vice versa, we arrive at the observational equations

$$\underline{Y}_P - \underline{y}_p = A_1\ \Delta\underline{x} \qquad 3(4)$$

$$\underline{Y}_P - \underline{y}_p = A_2\ \delta\underline{y}\ . \qquad 3(5)$$

These depend either on geometric or on gravimetric unknowns. Thus we have two alternatives in adjusting a geodetic network, a geometric mode or a gravimetric mode, as shown by the following examples.

EXAMPLE 3.1 (geometry space)

As shown in example 1.1, geometric positional angles or length ratios are independent of a reference system with degrees of freedom for translation, rotation, and scale. The linearized observational equations read

$$\underline{y}^{x}_{312} - \underline{y}^{x}_{312} = \Delta\underline{y}^{x}_{312} = \left[A_{11} \mid A_{12} \mid A_{13}\right] \left[\begin{array}{c} \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \\ \hline \Delta x_2 \\ \Delta y_2 \\ \Delta z_2 \\ \hline \Delta x_3 \\ \Delta y_3 \\ \Delta z_3 \end{array}\right] = A_1 \, \Delta\underline{x} \qquad 3(6)$$

where the matrices A_{1i} are functions of the coordinates (x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) at the points (p_1, p_2, p_3), and $\underline{y}^{x}_{312}$ represents either positional angles or length ratios. If horizontal directions and horizon distances have been observed, they are correlated observations, in general. (See formula 1(16).)

The vector $\Delta\underline{x}$ of geometric coordinate corrections can now be transformed into gravimetric coordinate disturbances by the three-dimensional Bruns formula, e.g., 2(31), 2(84) or 2(108).

(a) Isoparametric mapping of type longitude, latitude, geopotential:

$$\begin{bmatrix} \Delta x^{\cdot} \\ \Delta y^{\cdot} \\ \Delta z^{\cdot} \end{bmatrix} = B^{\cdot} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w \end{bmatrix} \qquad 3(7)$$

$$\delta\lambda = \left(\gamma_x^2 + \gamma_y^2\right)^{-1} \left(-\gamma_y \frac{\partial\delta w}{\partial x} + \gamma_x \frac{\partial\delta w}{\partial y}\right) \qquad 3(8)$$

$$\delta\phi = \gamma^{-2} \left(\gamma_x^2 + \gamma_y^2\right)^{-1/2} \left[-\gamma_x\gamma_z \frac{\partial\delta w}{\partial x} - \gamma_y\gamma_z \frac{\partial\delta w}{\partial y} + \left(\gamma_x^2 + \gamma_y^2\right) \frac{\partial\delta w}{\partial z}\right] \qquad 3(9)$$

(Grafarend (1978a): formulas 1(21), 1(22)).

$$\underline{y}^{x}_{312} - \underline{y}^{x}_{312} = \Delta\underline{y}^{x}_{312} = \left[A_{11}\ B_1 \mid A_{12}\ B_2 \mid A_{13}\ B_3\right] \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \\ \delta w_1 \\ \delta\lambda_2 \\ \delta\phi_2 \\ \delta w_2 \\ \delta\lambda_3 \\ \delta\phi_3 \\ \delta w_3 \end{bmatrix} = A_2\ \delta\underline{y} \quad . \qquad 3(10)$$

Thus, we have found that the observed geometric positional angles or distance ratios depend now only on the disturbing potential δw and the coordinates $\partial_i \delta w$ of its gradients. Equations 3(6) and 3(10) are dual.

If we know scale, distance observations can be approached in the same way.

(b) Isoparametric mapping of type longitude, latitude, gravity:

$$\begin{bmatrix} \Delta x^{\bullet} \\ \Delta y^{\bullet} \\ \Delta z^{\bullet} \end{bmatrix} = B^{\bullet} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta\gamma \end{bmatrix} \qquad 3(11)$$

$\delta\lambda$ 3(8)
$\delta\phi$ 3(9)

$$\delta\gamma = \gamma^{-1} \left(\gamma_x \frac{\partial \delta w}{\partial x} + \gamma_y \frac{\partial \delta w}{\partial y} + \gamma_z \frac{\partial \delta w}{\partial z} \right) \qquad 3(12)$$

(Grafarend (1978b): formulas (1.38)),

or

$$\begin{bmatrix} \Delta x^{\bullet} \\ \Delta y^{\bullet} \\ \Delta z^{\bullet} \end{bmatrix} = B^{\bullet} \begin{bmatrix} \delta\gamma_x \\ \delta\gamma_y \\ \delta\gamma_z \end{bmatrix} . \qquad 3(13)$$

($B^{\bullet}$ is, of course, a general notation, and not equal to $B^{\bullet}$ in 3(11).)

$$\delta\gamma_i = \partial_i \, \delta w \quad . \qquad 3(14)$$

$$\underline{y}^{x}_{312} - \underline{y}^{x}_{312} = \Delta\underline{y}^{x}_{312} = \left[A_{11} \; B_1 \mid A_{12} \; B_2 \mid A_{13} \; B_3 \right] \begin{bmatrix} \delta\gamma_{x_1} \\ \delta\gamma_{y_1} \\ \delta\gamma_{z_1} \\ \delta\gamma_{x_2} \\ \delta\gamma_{y_2} \\ \delta\gamma_{z_2} \\ \delta\gamma_{x_3} \\ \delta\gamma_{y_3} \\ \delta\gamma_{z_3} \end{bmatrix} = A_2 \, \delta\underline{y} \quad . \qquad 3(15)$$

Thus, we have found that the observed geometric positional angles or distance ratios depend now only on the coordinates $\partial_i \delta w$ of the gradient of the disturbing potential. If we know scale, distance observations can be approached in the same way.

(c) Isoparametric mapping of type longitude, latitude, gravity gradient:

$$\begin{bmatrix} \Delta x^{\bullet} \\ \Delta y^{\bullet} \\ \Delta z^{\bullet} \end{bmatrix} = B^{\bullet} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w_{ij} \end{bmatrix} \qquad 3(16)$$

$\delta\lambda$ 3(8)
$\delta\phi$ 3(9)

$$\delta w_{ij} = \partial_i \partial_j \, \delta w \qquad 3(17)$$

$$\underline{Y}^{x}_{312} - \underline{y}^{x}_{312} = \Delta \underline{y}^{x}_{312} = \left[A_{11} \; B_1 \mid A_{12} \; B_2 \mid A_{13} \; B_3 \right] = \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \\ \delta w^1_{ij} \\ \delta\lambda_2 \\ \delta\phi_2 \\ \delta w^2_{ij} \\ \delta\lambda_3 \\ \delta\phi_3 \\ \delta w^3_{ij} \end{bmatrix} = A_2 \; \delta \underline{y} \quad . \qquad 3(18)$$

Thus, we have found that the observed geometric positional angles or distance ratios depend now only on the coordinates $\partial_i \delta w$ and $\partial_i \partial_j \delta w$ of the first- and second-order gradients of the disturbing potential.

If we know scale, distance observations can be approached in the same way.

<u>EXAMPLE 3.2 (gravity space)</u>:

As we have seen in example 1.2 gravimetric positional angles or length ratios are independent of a reference system with degrees of freedom for translation, rotation, and scale. The linearized observational equations read

$$\underline{Y}^{\Gamma}_{312} - \underline{y}^{\Gamma}_{312} = \Delta\underline{Y}^{\Gamma}_{312} = \left[A_{21} \mid A_{22} \mid A_{23}\right] \begin{bmatrix} \delta\gamma_{x_1} \\ \delta\gamma_{y_1} \\ \delta\gamma_{z_1} \\ \delta\gamma_{x_2} \\ \delta\gamma_{y_2} \\ \delta\gamma_{z_2} \\ \delta\gamma_{x_3} \\ \delta\gamma_{y_3} \\ \delta\gamma_{z_3} \end{bmatrix} = A_2\ \delta\underline{y} \qquad 3(19)$$

where the matrices A_{2i} are functions of the coordinates (γ_{x_1}, γ_{y_1}, γ_{z_1}, γ_{x_2}, γ_{y_2}, γ_{z_2}, γ_{x_3}, γ_{y_3}, γ_{z_3}) at the points (γ_1, γ_2, γ_3) and Y^{Γ}_{312} represents either positional angles or length ratios in gravity space. If astronomical longitude, latitude, and gravity have been observed, they are correlated observations, in general. (See formula 1(24).)

The vector δy of gravimetric coordinate corrections can now be transformed into geometric coordinate corrections by the inverse of the three-dimensional Bruns formula, e.g., 2(84).

$$\begin{bmatrix} \delta\gamma_x \\ \delta\gamma_y \\ \delta\gamma_z \end{bmatrix} = C^{\cdot} \begin{bmatrix} \Delta x^{\cdot} \\ \Delta y^{\cdot} \\ \Delta z^{\cdot} \end{bmatrix} \qquad 3(20)$$

$$\underline{Y}^{\Gamma}_{312} - \underline{y}^{\Gamma}_{312} = \Delta\underline{y}^{\Gamma}_{312} = \left[A_{21}\ C_1 \mid A_{22}\ C_2 \mid A_{23}\ C_3\right] \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \\ \Delta x_2 \\ \Delta y_2 \\ \Delta z_2 \\ \Delta x_3 \\ \Delta y_3 \\ \Delta z_3 \end{bmatrix} = A_1 \Delta\underline{x} \quad . \qquad 3(21)$$

Thus, we have found that the observed gravimetric positional angles or length ratios depend now only on geometric coordinate corrections. Equations 3(15) and 3(21) are dual.

The next step is to assume that we know orientation and scale in gravity space. The linear observational equations for astronomical longitude and latitude, the length of the gravity vector, and potential differences read

$$\Lambda_P - \lambda_p = (\partial\lambda/\partial x)_p \, \Delta x + (\partial\lambda/\partial y)_p \, \Delta y + (\partial\lambda/\partial z)_p \, \Delta z + \delta\lambda \qquad 3(22)$$

$$\Phi_P - \phi_p = (\partial\phi/\partial x)_p \, \Delta x + (\partial\phi/\partial y)_p \, \Delta y + (\partial\phi/\partial z)_p \, \Delta z + \delta\phi \qquad 3(23)$$

$$\Gamma_P - \gamma_p = (\partial\gamma/\partial z)_p \, \Delta x + (\partial\gamma/\partial y)_p \, \Delta y + (\partial\gamma/\partial z)_p \, \Delta z + \delta\gamma \qquad 3(24)$$

$$\begin{aligned} W_2 - W_1 - (w_2 - w_1) = & - (\partial w/\partial x)_{P_1} \Delta x^1 - (\partial w/\partial y)_{P_1} \Delta y^1 - (\partial w/\partial z)_{P_1} \Delta z^1 \\ & + (\partial w/\partial z)_{P_2} \Delta x^2 + (\partial w/\partial y)_{P_2} \Delta y^2 + (\partial w/\partial z)_{P_2} \Delta z^2 \\ & + \delta w_2 - \delta w_1 . \end{aligned} \qquad 3(25)$$

(a) Isoparametric mapping of type longitude, latitude, geopotential:

$$\begin{bmatrix} \Delta x^{\cdot} \\ \Delta y^{\cdot} \\ \Delta z^{\cdot} \end{bmatrix} = B^{\cdot} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w \end{bmatrix} \qquad 3(26)$$

$\delta\lambda$ 3(8)
$\delta\phi$ 3(9)
$\delta\gamma$ 3(12)

$$\Lambda_P - \lambda_p = \Delta\lambda = 0$$
$$\Phi_P - \phi_p = \Delta\phi = 0$$
$$W_P - w_p = \Delta w = 0$$

$$2\ \delta w + \rho\ \frac{\partial \delta w}{\partial \rho} = \frac{\Delta\gamma}{\gamma_p}\ \rho\ \frac{\partial w}{\partial \rho} \qquad 3(27)$$

where

$$\rho = \gamma^{-1/2}$$

(Grafarend (1978a): formula 1(23)).

The observational equation for the length of the gravity vector is well known; it is the boundary condition for the "harmonic" potential δw. Even better known is its zero-order approximation based on 2(21)

$$\frac{2}{r}\ \delta w + \frac{\partial}{\partial r}\ (\delta w) = -\ \Delta\gamma \qquad 3(28)$$

(Grafarend (1978a): formula 1(26)).

Thus, we have found that the observed length of the gravity vector depends only on the disturbing potential δw and the coordinate $\partial\delta w/\partial\rho$ or $\partial\delta w/\partial r$ of its gradient vector, where $r = ||\underline{x}||$.

The dual formulation is obtained if we make use of the inverse three-dimensional Bruns transformation

$$\delta w = C_3\ \Delta\underline{x} \quad . \qquad 3(29)$$

Employing the summation convention over i = x,y,z,

$$\delta\underline{\gamma} = \gamma^{-1} C_3 \gamma_i \partial_i \Delta\underline{x} \qquad 3(30)$$

$$\underline{\Gamma}_P - \underline{\gamma}_p = \Delta\underline{\gamma} = A_\gamma \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} + \gamma^{-1} C_3 \gamma_i \partial_i \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \qquad 3(31)$$

where the vector of geometric coordinate corrections $\Delta\underline{x}$ is a "harmonic" function as long as we measure in empty space. Again we have found that the observed length of the gravity vector depends only on the geometry vector $\Delta\underline{x}$ and its gradient. Equations 3(27) (or 3(28)) and 3(31) are dual.

(b) Isoparametric mapping of type longitude, latitude, gravity

$$\begin{bmatrix} \Delta x^{\cdot} \\ \Delta y^{\cdot} \\ \Delta z^{\cdot} \end{bmatrix} = B^{\cdot} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta\gamma \end{bmatrix} \qquad 3(32)$$

$\delta\lambda$ 3(8)

$\delta\phi$ 3(9)

$\delta\gamma$ 3(12)

$$\Lambda_P - \lambda_p = \Delta\lambda = 0$$
$$\Phi_P - \phi_p = \Delta\phi = 0$$
$$\Gamma_P - \gamma_p = \Delta\gamma = 0$$

$$W_P - w_p = \Delta w = \delta w + \frac{\rho}{2} \frac{\partial \delta w}{\partial \rho} \qquad 3(33)$$

where

$$\rho = \gamma^{-1/2}$$

(Grafarend (1978b): formula (1.39)).

The observational equation for the potential is well known; it serves alternatively as the boundary condition for the "harmonic" potential δw. For the zero-order approximation 2(21) we find

$$\delta w + \frac{r}{2}\frac{\partial}{\partial r}(\delta w) = \Delta w \quad . \qquad 3(34)$$

Thus, we have found that the absolute potential depends only on the disturbing potential δw and the coordinate $\partial\delta w/\partial\rho$ or $\partial\delta w/\partial r$ of its gradient vector where

$$r = ||\underline{x}|| \quad .$$

For the dual formulation we mention only that we have to integrate $\delta\gamma$ in order to arrive at δw.

(c) Isoparametric mapping of type longitude, latitude, and gravity gradient

$$\begin{bmatrix} \Delta x^{\cdot} \\ \Delta y^{\cdot} \\ \Delta z^{\cdot} \end{bmatrix} = B^{\cdot} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w_{ij} \end{bmatrix} \qquad 3(35)$$

$\delta\lambda$ 3(8)

$\delta\phi$ 3(9)

δw_{ij} 3(17)

$$\Lambda_P - \lambda_p = \Delta\lambda = 0$$
$$\Phi_P - \phi_p = \Delta\phi = 0$$
$$W_{ij}^{P} - w_{ij}^{p} = \Delta w_{ij} = 0$$

$$W_P - w_p = (\text{grad } w)_p \, B^{\cdot} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w_{ij} \end{bmatrix} + \delta w \qquad 3(36)$$

$$\Gamma_P - \gamma_p = (\text{grad } \gamma)_p \, B^{\cdot} \begin{bmatrix} \delta\lambda \\ \delta\phi \\ \delta w_{ij} \end{bmatrix} + \delta\gamma \quad . \qquad 3(37)$$

Thus, we have found that the observed potential and length of the gravity vector depend only on the disturbing potential and coordinates of its first- and second-order gradient.

Another set of geodetic observations depends on the gravity field, e.g., astronomical azimuth A and horizon distance B. Their observational equations are structured according to

$$\begin{aligned} A_{12} - \alpha_{12} = {} & (\partial\alpha_{12}/\partial x^1)\Delta x^1 + (\partial\alpha_{12}/\partial y^1)\Delta y^1 + (\partial\alpha_{12}/\partial z^1)\Delta z^1 \\ & + (\partial\alpha_{12}/\partial x^2)\Delta x^2 + (\partial\alpha_{12}/\partial y^2)\Delta y^2 + (\partial\alpha_{12}/\partial z^2)\Delta z^2 + \delta\alpha_{12} \end{aligned} \qquad 3(38)$$

$$\begin{aligned} B_{12} - \beta_{12} = {} & (\partial\beta_{12}/\partial x^1)\Delta x^1 + (\partial\beta_{12}/\partial y^1)\Delta y^1 + (\partial\beta_{12}/\partial z^1)\Delta z^1 \\ & + (\partial\beta_{12}/\partial x^2)\Delta x^2 + (\partial\beta_{12}/\partial y^2)\Delta y^2 + (\partial\beta_{12}/\partial z^2)\Delta z^2 + \delta\beta_{12} \end{aligned} \qquad 3(39)$$

where α, β are zero-order approximations of A, B and $\delta\alpha$. $\delta\beta$ are subject to the Laplace condition.

$$\begin{bmatrix} \delta\alpha \\ \delta\beta \end{bmatrix} = L \begin{bmatrix} \delta\lambda \\ \delta\phi \end{bmatrix} \qquad 3(40)$$

Here, we assumed that the base vectors $F^{\cdot}$ and $f^{\cdot}$ are parallel in the Euclidean sense. Hence, the observational equations will read

$$\underline{Y}_{12} - \underline{y}_{12} = \Delta\underline{y}_{12} = A_y \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \\ \Delta x_2 \\ \Delta y_2 \\ \Delta z_2 \end{bmatrix} + L \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \end{bmatrix} \quad . \qquad 3(41)$$

The equations can be formulated in the geometric mode if we transform $\delta\lambda$, $\delta\phi$ according to the inverse three-dimensional Bruns formula of types (a), (b) or (c). In all cases we will obtain

$$\Delta\underline{y}_{12} = A_2 \, \Delta\underline{x} \quad . \qquad 3(42)$$

The dual representation is found to be

$$\Delta\underline{y}_{12} = A_1 \, \delta\underline{y} \qquad 3(43)$$

where

$$\text{(i)} \quad \delta\underline{y} = \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \\ \delta w_1 \\ \delta\lambda_2 \\ \delta\phi_2 \\ \delta w_2 \end{bmatrix} \qquad \text{(ii)} \quad \delta\underline{y} = \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \\ \delta\gamma_1 \\ \delta\lambda_2 \\ \delta\phi_2 \\ \delta\gamma_2 \end{bmatrix} \qquad \text{(iii)} \quad \delta\underline{y} = \begin{bmatrix} \delta\lambda_1 \\ \delta\phi_1 \\ \delta w_{ij}{}_1 \\ \delta\lambda_2 \\ \delta\phi_2 \\ \delta w_{ij}{}_2 \end{bmatrix} \qquad 3(44)$$

if we choose different isoparametric mappings.

At this point we introduce the unknowns describing the gravity field of the Earth, for instance, the mass density virials

$$I_o, \; I_i, \; I_{ij}, \; I_{ijk}, \; \ldots$$

given in the appendix. They appear in the form

$$I_o \, ||x||^{-1}, \; I_i \, x_i \, ||x||^{-3}, \; I_{ij} \, x_i \, x_j \, ||x||^{-5}, \; \ldots \qquad 3(45)$$

and must fulfill the Laplace differential equation. Because of this condition, the degree of freedom is 2n+1, where n is the order of approximation in a series for the gravitational potential, e.g., for n = 2 only 5 coefficients of the six

$$I_{11}, \; I_{22}, \; I_{12}, \; I_{13}, \; I_{23} \qquad 3(46)$$

are independent as a result of tr $I_{ij} = 0$.

The quantities "observed minus computed" $\underline{Y}_P - \underline{y}_p$ are represented by

$$\begin{aligned} \underline{Y}_P - \underline{y}_p = {} & (\partial \underline{y}/\partial x)_p \Delta x + (\partial \underline{y}/\partial y)_p \Delta y + (\partial \underline{y}/\partial z)_p \Delta z \\ & + (\partial \underline{y}/\partial I_o)_p \, \delta I_o \\ & + (\partial \underline{y}/\partial I_1)_p \, \delta I_1 + (\partial \underline{y}/\partial I_2)_p \, \delta I_2 + (\partial \underline{y}/\partial I_3)_p \, \delta I_3 \\ & + (\partial \underline{y}/\partial I_{11})_p \, \delta I_{11} \\ & + (\partial \underline{y}/\partial I_{22})_p \, \delta I_{22} \\ & + (\partial \underline{y}/\partial I_{12})_p \, \delta I_{12} \\ & + (\partial \underline{y}/\partial I_{13})_p \, \delta I_{13} \\ & + (\partial \underline{y}/\partial I_{23})_p \, \delta I_{23} \\ & + \ldots + \delta \underline{y} \quad . \end{aligned} \qquad 3(47)$$

By the symbol "p" we understand the origin of the Taylor series, a set of approximate coordinates of points in the geometry space and of approximate gravity field parameters. How can these be determined?

To answer this question we present an example.

Let us introduce the isoparametric mapping

$$\begin{bmatrix} \Delta\lambda & = 0 \\ \Delta\phi & = 0 \\ \Delta w & = 0 \\ \Delta\gamma & = 0 \end{bmatrix} \qquad 3(48)$$

applied to a zero-order approximation

$$w = - gm \, ||\mathbf{x}||^{-1}$$

of the gravity field.

$$\Lambda = \text{arc tan} \frac{y}{x}$$

$$\Phi = \text{arc tan} \frac{z}{\sqrt{x^2 + y^2}}$$

$$W = - \frac{gm}{\sqrt{x^2 + y^2 + z^2}}$$

$$\Gamma = \frac{gm}{x^2 + y^2 + z^2}$$

are the global mapping equations. The general solution can be represented by

$$x = ||\underline{x}|| \cos \Lambda \cos \Phi$$

$$y = ||\underline{x}|| \sin \Lambda \cos \Phi$$

$$z = ||\underline{x}|| \sin \Phi$$

$$||\underline{x}|| = -\frac{gm}{W} = \sqrt{\frac{gm}{\Gamma}} \qquad 3(49)$$

$$gm = \frac{W^2}{\Gamma} \quad . \qquad 3(50)$$

Thus we have derived approximate values for (x, y, z) and gm:

$$x = \frac{W}{\Gamma} \cos \Lambda \cos \Phi \qquad 3(51)$$

$$y = \frac{W}{\Gamma} \sin \Lambda \cos \Phi \qquad 3(52)$$

$$z = \frac{W}{\Gamma} \sin \Phi \qquad 3(53)$$

$$gm = \frac{W^2}{\Gamma} \quad . \qquad 3(54)$$

ACKNOWLEDGMENT

I wish to express my appreciation to the National Geodetic Survey (NGS) staff for their warm hospitality. Inspiring discussions were held with John D. Bossler, Bernard Chovitz, and John G. Gergen of NGS, Hyman O. Orlin of the National Academy of Sciences, and last, but not least, Petr Vaníček, Visiting Senior Scientist in Geodesy of the National Research Council. Their cooperation is gratefully acknowledged.

APPENDIX A.--INVARIANT REPRESENTATION OF $||x - x'||^{-m}$

In terms of Hilbert invariants

$$i = ||\underline{x}|| \tag{A1}$$

$$i' = ||\underline{x}'|| \tag{A2}$$

$$i'' = \frac{(\underline{x}, \underline{x}')}{||\underline{x}||\ ||\underline{x}'||} \tag{A3}$$

we will set up series for

$$||\underline{x} - \underline{x}'||^{-m} = i^{-m}\left[1 + \frac{i'^2}{i^2} - 2\,\frac{i'}{i}\,i''\right]^{\frac{m}{2}} \quad . \tag{A4}$$

Set

$$\varepsilon = \frac{i'^2}{i^2} - 2\,\frac{i'}{i}\,i'' \tag{A5}$$

$$\begin{aligned}(1+\varepsilon)^{-\frac{m}{2}} &= 1 - \frac{m}{1!\ 2}\,\varepsilon + \frac{m(m+2)}{2!\ 4}\,\varepsilon^2 \\ &\quad - \frac{m(m+2)(m+4)}{3!\ 8}\,\varepsilon^3 + \frac{m(m+2)(m+4)(m+6)}{4!\ 16}\,\varepsilon^4 \\ &\quad - o(\varepsilon^5) \\ &= \sum_{n=o}^{\infty} (-1)^n\,\frac{m(m+2)(m+4)\ldots(m+2n-2)}{n!\ 2^n}\,\varepsilon^n \quad .\end{aligned} \tag{A6}$$

$i'' \leq 1$; thus the series are convergent within the sphere determined by $\frac{||\underline{x}||}{||\underline{x}||} < 1$. Under this assumption we can represent (A6) by

$$||\underline{x}-\underline{x}'||^{-m} = \sum_{n=o}^{\infty} i'^{n}\, i^{-n-m}\, P_n^m(i'') \tag{A7}$$

where $P_n^m(i'')$ are Gegenbauer (ultraspherical) polynomials (Abramowitz and Stegun, 1964: p. 774)

$$P_o^m(x) = 1 \tag{A8}$$

$$P_1^m(x) = mx \tag{A9}$$

$$P_2^m(x) = \frac{m}{2}\,[(m+2)x^2 - 1] \tag{A10}$$

$$P_3^m(x) = \frac{m(m+2)}{6}\, x\, [(m+4)x^2 - 3] \tag{A11}$$

$$P_4^m(x) = \frac{m(m+2)}{24}\, [(m+4)(m+6)x^4 - 6\,(m+4)x^2 + 3] \tag{A12}$$

...

$$P_n^m(x) = \frac{1}{n!}\, \Big\{ m(m+2)\ldots(m+2n-2)x^n$$

$$- \frac{n(n-1)}{2}\, m(m+2)\ldots(m+2n-4)x^{n-2}$$

(A13)

$$+ \frac{n(n-1)(n-2)(n-3)}{2\cdot 4} m(m+2)\ldots(m+2n-6)x^{n-4}$$

$$- \ldots$$

$$+ (-1)^{\ell} \frac{n(n-1)\cdots(n-2\ell+1)}{2\cdot 4\cdots 2\ell} m(m+2)\ldots[m+2(n-\ell-1)]\ x^{n-2\ell}\Big\} ,$$

$$\ell = 0,1,\cdots \leq \frac{n}{2} \quad .$$

Despite the identical notation, these should not be confused with associated Legendre polynomials.

APPENDIX B.--CARTESIAN REPRESENTATION OF THE GRAVITATIONAL POTENTIAL

In the appendices B-E, we change to index notation (x_1, x_2, x_3) in place of (x, y, z) and utilize the summation convention.

Let

$$u(\underline{x}) = - g \int dx' \, \rho(\underline{x}') \, ||\underline{x}-\underline{x}'||^{-1} \tag{B1}$$

be the Newtonian representation of the gravitational potential. In terms of Hilbert invariants

$$i = ||\underline{x}|| \tag{B2}$$

$$i' = ||\underline{x}'|| \tag{B3}$$

$$i'' = \frac{(\underline{x},\underline{x}')}{||\underline{x}|| \; ||\underline{x}'||} \tag{B4}$$

$$||\underline{x}-\underline{x}'||^{-1} = \sum_{n=o}^{\infty} i'^{n} \, i^{-n-1} \, P_n(i'') \tag{B5}$$

holds if $\frac{||\underline{x}'||}{||\underline{x}||} < 1$. Under this assumption the gravitational potential can be represented by

$$u(\underline{x}) = -\sum_{n=o}^{\infty} g \, i^{-n-1} \int dx' i'^{n} \, P_n(i'') \, \rho(\underline{x}') \tag{B6}$$

where P_n are the standard Legendre polynomials.

In cartesian coordinates the Hilbert invariants read

$$i = ||\underline{x}|| = \sqrt{x_i x_i} \tag{B7}$$

$$i' = ||\underline{x}'|| = \sqrt{x_i' x_i'} \tag{B8}$$

$$i'' = \frac{x_i x_i'}{\sqrt{x_i x_i}\,\sqrt{x_i' x_i'}} \tag{B9}$$

and the related Legendre polynomials $P_n(i'')$

$$P_o(i'') = 1 \tag{B10}$$

$$P_1(i'') = x_i x_i' \div \left(\sqrt{x_j x_j}\,\sqrt{x_k' x_k'}\right) \tag{B11}$$

$$P_2(i'') = \frac{3}{2}\,\frac{x_i x_i'\; x_j x_j' - \frac{1}{3}\, x_p x_p\; x_q' x_q'}{x_k x_k\; x_\ell' x_\ell'} \tag{B12}$$

$$P_3(i'') = \frac{5}{2}\,\frac{x_i x_i'}{\sqrt{x_j x_j}\,\sqrt{x_k' x_k'}}\;\frac{x_\ell x_\ell'\; x_m x_m' - \frac{3}{5}\, x_r x_r\; x_s' x_s'}{x_p x_p\; x_q' x_q'} \tag{B13}$$

$$P_4(i'') = \frac{35}{8}\,\frac{(x_i x_i')^4 - \frac{6}{7}\,(x_i x_i')(x_p x_p)(x_q' x_q') + \frac{3}{35}\,(x_r x_r)^2 (x_s' x_s')^2}{(x_j x_j)^2 (x_k' x_k')^2} \tag{B14}$$

etc.

(B6) can be written

In cartesian coordinates the Hilbert invariants read

$$i = ||\underline{x}|| = \sqrt{x_i x_i} \tag{B7}$$

$$i' = ||\underline{x}'|| = \sqrt{x_i' x_i'} \tag{B8}$$

$$i'' = \frac{x_i x_i'}{\sqrt{x_i x_i}\,\sqrt{x_i' x_i'}} \tag{B9}$$

and the related Legendre polynomials $P_n(i'')$

$$P_o(i'') = 1 \tag{B10}$$

$$P_1(i'') = x_i x_i' \div \left(\sqrt{x_j x_j}\,\sqrt{x_k' x_k'}\right) \tag{B11}$$

$$P_2(i'') = \frac{3}{2}\,\frac{x_i x_i'\; x_j x_j' - \frac{1}{3}\, x_p x_p\; x_q' x_q'}{x_k x_k\; x_\ell' x_\ell'} \tag{B12}$$

$$P_3(i'') = \frac{5}{2}\,\frac{x_i x_i'}{\sqrt{x_j x_j}\,\sqrt{x_k' x_k'}}\;\frac{x_\ell x_\ell'\; x_m x_m' - \frac{3}{5}\, x_r x_r\; x_s' x_s'}{x_p x_p\; x_q' x_q'} \tag{B13}$$

$$P_4(i'') = \frac{35}{8}\,\frac{(x_i x_i')^4 - \frac{6}{7}\,(x_i x_i')(x_p x_p)(x_q' x_q') + \frac{3}{35}\,(x_r x_r)^2 (x_s' x_s')^2}{(x_j x_j)^2 (x_k' x_k')^2} \tag{B14}$$

etc.

(B6) can be written

$$u(x) = \sum_{n=o}^{\infty} u_n \tag{B15}$$

where

$$u_o = -\frac{gm}{\sqrt{x_i x_i}} = -\frac{g}{(x_i x_i)^{1/2}} I_o \tag{B16}$$

$$u_1 = -\frac{gx_i}{(x_j x_j)^{3/2}} \int dm' x_i' = -\frac{g}{(x_j x_j)^{3/2}} x_i I_i \tag{B17}$$

$$u_2 = -\frac{3g}{2(x_k x_k)^{5/2}} x_i x_j \int dm' x_i' x_j' + \frac{g}{2(x_k x_k)^{3/2}} dm' x_i' x_i' \tag{B18}$$

$$= -\frac{3g}{2(x_k x_k)^{5/2}} x_i x_j I_{ij}$$

$$u_3 = -\frac{5g}{2(x_\ell x_\ell)^{7/2}} x_i x_j x_k \int dm' x_i' x_j' x_k' \tag{B19}$$

$$+ \frac{3g}{2(x_\ell x_\ell)^{3/2}} x_i \delta_{jk} =$$

$$- \frac{5g}{2(x_\ell x_\ell)^{7/2}} x_i x_j x_k \; I_{ijk}$$

$$u_4 = -\frac{35g}{8(x_n x_n)^{9/2}} x_i x_j x_k x_\ell \int dm' \left[x_i' x_j' x_k' x_\ell' - \frac{6}{7} x_i' x_j' \delta_{k\ell} (x_m' x_m')\right.$$

$$\left. + \frac{3}{35} \delta_{ij} \delta_{k\ell} (x_m' x_m')^2\right] = -\frac{35g}{8(x_m x_m)^{9/2}} x_i x_j x_k x_\ell I_{ijk\ell} \tag{B20}$$

and

$$I_o = \int dm = m \tag{B21}$$

$$I_i = \int dm\, x_i \tag{B22}$$

$$I_{ij} = \int dm \left[x_i x_j - \frac{1}{3} (x_k x_k) \delta_{ij}\right] \tag{B23}$$

$$I_{ijk} = \int dm \left[x_i x_j x_k - \frac{3}{5} (x_\ell x_\ell) x_i \delta_{jk}\right] \tag{B24}$$

$$I_{ijk\ell} = \int dm \left[x_i x_j x_k x_\ell - \frac{6}{7} x_i x_j \delta_{k\ell} (x_m x_m) + \frac{3}{35} \delta_{ij} \delta_{k\ell} (x_m x_m)^2\right] \tag{B25}$$

etc.

In summarizing, we can represent the gravitational potential in Cartesian coordinates by

$$u(x) = -\frac{gm}{\sqrt{x_j x_j}} - \sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)}{n!\ (x_j x_j)^{(2n+1)/2}} g x_{i_1} x_{i_2} \ldots x_{i_n} I_{i_1 i_2 \ldots i_n} \tag{B26}$$

where

$$I_{i_1 i_2 \ldots i_n} = \int dm \quad x_{i_1} x_{i_2} \ldots x_{i_n} - \frac{n(n-1)}{2(2n-1)} x_{i_1} x_{i_2} \ldots x_{i_{n-2}} \delta_{i_{n-1} i_n} x_k x_k \tag{B27}$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-3)} x_{i_1} x_{i_2} \ldots x_{i_{n-4}} \delta_{i_{n-3} i_{n-2}} \delta_{i_{n-1} i_n} (x_k x_k)^2$$

$$- \ldots + (-1)^{\ell} \frac{n(n-1)\ldots(n-2\ell+1)}{2\cdot 4\ldots 2\ell(2n-1)(2n-3)\ldots(2n-2\ell+1)}$$

$$\left[\left(x_{i_1} x_{i_2} \ldots x_{i_{n-2\ell}} \delta_{i_{n-(2\ell+1)} i_{n-(2\ell+2)}} \ldots \delta_{i_{n-1} i_n} (x_k x_k)^{\ell}\right)\right],$$

$$\ell = 0, 1, \ldots, \leq \frac{n}{2} .$$

APPENDIX C.--CARTESIAN REPRESENTATION OF FIRST-ORDER GRADIENTS OF THE GRAVITATIONAL POTENTIAL

Let

$$\partial u/\partial x_i = + g \int dx' \rho(\underline{x}') \; (x_i - x_i') \; ||\underline{x} - \underline{x}'||^{-3} \tag{C1}$$

be the Newtonian representation of the first-order gradients of the gravitational potential. In terms of Hilbert invariants

$$i = ||\underline{x}|| \tag{C2}$$

$$i' = ||\underline{x}'|| \tag{C3}$$

$$i'' = \frac{(\underline{x},\underline{x}')}{||\underline{x}|| \; ||\underline{x}'||} \tag{C4}$$

$$||\underline{x} - \underline{x}'||^{-3} = \sum_{n=o}^{\infty} i'^n \; i^{-n-3} \; P_n^3 \; (i'') \tag{C5}$$

holds if $\frac{||x'||}{||x||} < 1$. Under this assumption the first-order gradients of the gravitational potential can be represented by

$$\partial u/\partial x_i = + \sum_{n=o}^{\infty} g \; i^{-n-3} \int dx' i'^n \; P_n^3(i'') \; (x_i - x_i') \; \rho(x') \tag{C6}$$

where P_n^3 are Gegenbauer polynomials (see (A8) - (A13))

(C7)

$$\begin{cases} P_0^3(x) = 1 \\ P_1^3(x) = 3x \\ P_2^3(x) = \frac{3}{2}(5x^2-1) \\ P_3^3(x) = \frac{5}{2}x(7x^2-3) \\ P_4^3(x) = \frac{15}{8}(21x^4-14x^2+1) \end{cases}$$

...

(C8)

$$P_n^3(x) = \frac{1}{n!}\Bigg[3\cdot 5\ldots(3+2n-2)x^n$$

$$- \frac{n(n-1)}{2}\, 3\cdot 5\ldots(3+2n-4)x^{n-2}$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2\cdot 4}\, 3\cdot 5\ldots(3+2n-6)x^{n-4}$$

$$- \ldots$$

$$+ (-1)^{\ell}\, \frac{n(n-1)\ldots(n-2\ell+1)}{2\cdot 4\ldots 2\ell}\, 3\cdot 5\ldots(2n-2\ell+1)x^{n-2\ell}\Bigg] ,$$

$$\ell = 0,1,\ldots < \frac{n}{2} .$$

If we refer to Equations (B15) to (B27) as the Cartesian representation of the gravitational potential, we will arrive at

$$\partial u_o/\partial x_i = + g \, ||\underline{x}||^{-3} \, x_i \, I_o \tag{C9}$$

$$\partial u_1/\partial x_i = - g \, ||\underline{x}||^{-3} \, I_i + 3g \, ||\underline{x}||^{-5} \, x_i x_j \, I_j \tag{C10}$$

$$\partial u_2/\partial x_i = + 3g \, ||\underline{x}||^{-5} \, x_j \, I_{ij} - \frac{15}{2} g \, ||\underline{x}||^{-7} \, x_i x_j x_k \, I_{jk} \tag{C11}$$

$$\partial u_3/\partial x_i = - \frac{15}{2} g \, ||\underline{x}||^{-7} \, x_j x_k \, I_{ijk} + \frac{35}{2} g \, ||\underline{x}||^{-9} \, x_i x_j x_k x_\ell \, I_{jk\ell} \tag{C12}$$

$$\partial u_4/\partial x_i = - \frac{35}{2} g \, ||\underline{x}||^{-9} \, x_j x_k x_\ell \, I_{ijk\ell} + \frac{315}{8} g \, ||\underline{x}||^{-11} \, x_i x_j x_k x_\ell x_m \, I_{jk\ell m} \tag{C13}$$

or, in general,

$$\begin{aligned} \partial u/\partial x_i = {} & + g \, ||\underline{x}||^{-3} \, x_i \, I_o \\ & - \sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)n}{n!} g \, ||\underline{x}||^{-(2n+1)} \, x_{i_1} \ldots x_{i_{n-1}} \, I_{ii_1 \ldots i_{n-1}} \\ & + \sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)(2n+1)}{n!} g \, ||\underline{x}||^{-(2n+3)} \, x_i x_{i_1} \ldots x_{i_n} \, I_{i_1 \ldots i_n} \end{aligned} \tag{C14}$$

where

$$||\underline{x}|| = \sqrt{x_k x_k} \quad .$$

APPENDIX D.--CARTESIAN REPRESENTATION OF SECOND-ORDER GRADIENTS OF THE GRAVITATIONAL POTENTIAL

We refer to equations (C9) to (C14) as the Cartesian representation of the first-order gradients of the gravitational potential. Here we will compute second-order gradients.

$$\partial^2 u_o/\partial x_i \partial x_j = -\ 3g\ ||\underline{x}||^{-5}\ I_o \left[x_i x_j - \frac{1}{3}\ ||\underline{x}||^2\ \delta_{ij}\right] \tag{D1}$$

$$\partial^2 u_1/\partial x_i \partial x_j = +\ 3g\ ||\underline{x}||^{-5}\ (x_i\ I_j + x_j\ I_i) \tag{D2}$$

$$-\ 15g\ ||\underline{x}||^{-7}\ x_k\ I_k \left[x_i x_j - \frac{1}{5}\ ||\underline{x}||^2\ \delta_{ij}\right]$$

$$\partial^2 u_2/\partial x_i \partial x_j = 3g\ ||\underline{x}||^{-5}\ I_{ij} \tag{D3}$$

$$-\ 15g\ ||\underline{x}||^{-7}\ x_k\ (x_j\ I_{ik} + x_i\ I_{jk})$$

$$+\ \frac{105}{2}\ g\ ||\underline{x}||^{-9}\ x_k x_\ell\ I_{k\ell} \left[x_i x_j - \frac{1}{7}\ ||\underline{x}||^2\ \delta_{ij}\right]$$

$$\partial^2 u_3/\partial x_i \partial x_j = -\ 15g\ ||\underline{x}||^{-7}\ x_k\ I_{ikj} \tag{D4}$$

$$+\ \frac{105}{2}\ g\ ||\underline{x}||^{-9}\ x_k x_\ell\ (x_i\ I_{jk\ell} + x_j\ I_{ik\ell})$$

$$-\ \frac{315}{2}\ g\ ||\underline{x}||^{-11}\ x_k x_\ell x_m\ I_{k\ell m} \left[x_i x_j - \frac{1}{9}\ ||\underline{x}||^2\ \delta_{ij}\right]$$

$$\partial^2 u_4/\partial x_i \partial x_j = -\frac{105}{2} g \, ||\underline{x}||^{-9} \, x_k x_\ell \, I_{ijk\ell}$$

$$+ \frac{315}{2} g \, ||\underline{x}||^{-11} \, x_k x_\ell x_m \, (x_i \, I_{jk\ell m} + x_j \, I_{ik\ell m}) \tag{D5}$$

$$- \frac{3465}{8} g \, ||\underline{x}||^{-13} \, x_k x_\ell x_m x_p \, I_{k\ell mp} \left[x_i x_j - \frac{1}{11} ||\underline{x}||^2 \, \delta_{ij}\right]$$

or, in general,

$$\partial^2 u/\partial x_i \partial x_j = -3g \, ||\underline{x}||^{-5} \, I_o \left[x_i x_j - \frac{1}{3} ||\underline{x}||^2 \, \delta_{ij}\right] \tag{D6}$$

$$-\sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)(n-1)n}{n!} g \, ||\underline{x}||^{-(2n+1)} \, I_{iji_2 \ldots i_{n-1}}$$

$$+\sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)(2n+1)n}{n!} g \, ||\underline{x}||^{-(2n+3)}$$

$$\cdot \, (x_i \, I_{ji_2 \ldots i_n} + x_j \, I_{ii_2 \ldots i_n})$$

$$+\sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots (2n-1)(2n+1)(2n+3)}{n!} g \, ||\underline{x}||^{-(2n+5)}$$

$$\cdot \left(x_{i_1} \ldots x_{i_n} \, I_{i_1 \ldots i_{n-1}}\right)\left(x_i x_j - \frac{1}{2n+3} ||\underline{x}||^2 \, \delta_{ij}\right)$$

where $||\underline{x}|| = \sqrt{x_k x_k}$. It is a "nice" exercise to prove

$$\mathrm{tr}\left(\partial^2 u_o/\partial x_i \partial x_j\right) = 0,\ \mathrm{tr}\left(\partial^2 u_1/\partial x_i \partial x_j\right) = 0,\ \mathrm{tr}\left(\partial^2 u_2/\partial x_i \partial x_j\right) = 0,$$

$$\mathrm{tr}\left(\partial^2 u_3/\partial x_i \partial x_j\right) = 0,\ \mathrm{tr}\left(\partial^2 u_4/\partial x_i \partial x_j\right) = 0,\ \text{or, in general, } \mathrm{tr}\left(\partial^2 u/\partial x_i \partial x_j\right) = 0.$$

APPENDIX E.--CARTESIAN REPRESENTATION OF THE EUCLIDEAN NORM OF FIRST-ORDER GRADIENTS OF THE GRAVITY POTENTIAL

Let $w = u - \frac{1}{2}\omega^2(x_1{}^2 + x_2{}^2)$ be the scalar part of the gravity potential such that $\gamma^2 = (\partial w/\partial x_i)(\partial w/\partial x_i)$ is the square of the Euclidean norm of its first-order gradients. We refer to (C14) as the Cartesian representation of the gravitational potential. The formula

$$\begin{aligned}\frac{\partial w}{\partial x_i} = {} & + g\,||\underline{x}||^{-3}\,x_i\,I_o \\ & -\sum_{n=1}^{\infty}\frac{1\cdot 3\ldots(2n-1)n}{n!}\,g\,||\underline{x}||^{-(2n+1)}\,x_{i_1}\ldots x_{i_{n-1}}\,I_{ii_1\ldots i_{n-1}} \\ & +\sum_{n=1}^{\infty}\frac{1\cdot 3\ldots(2n-1)(2n+1)}{n!}\,g\,||\underline{x}||^{-(2n+3)}\,x_i x_{i_1}\ldots x_{i_n}\,I_{i_1\ldots i_n} \\ & -\omega^2\,x_\alpha\,\delta_{\alpha i}\end{aligned} \tag{E1}$$

(where Greek indices range over 1, 2 only)

contains four terms and we designate it as a + b + c + d. The four-term scheme leads to

$$\begin{aligned}\gamma^2 = {} & a^2 + b^2 + c^2 + d^2 \\ & + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd.\end{aligned} \tag{E2}$$

Explicitly, it has the form

$$\gamma^2 = g^2 \, ||\underline{x}||^{-4} \, I_o^{\,2} \tag{E3}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)n}{n!} \frac{1\cdot 3\ldots(2m-1)m}{m!} \, g^2$$

$$||\underline{x}||^{-(2n+1)} \quad ||\underline{x}||^{-(2m+1)} \quad x_{i_1}\ldots x_{i_{n-1}} \; x_{j_1}\ldots x_{j_{m-1}}$$

$$\cdot \Big(I_{1i_1\ldots i_{n-1}} \, I_{1j_1\ldots j_{m-1}} + I_{2i_1\ldots i_{n-1}} \, I_{2j_1\ldots j_{m-1}}$$

$$+ I_{3i_1\ldots i_{n-1}} \, I_{3j_1\ldots j_{m-1}} \Big)$$

$$+ g^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)(2n+1)}{n!} \frac{1\cdot 3\ldots(2m-1)(2m+1)}{m!}$$

$$||\underline{x}||^{-(2n+2)} \quad ||\underline{x}||^{-(2m+2)} \quad x_{i_1}\ldots x_{i_n} \; x_{j_1}\ldots x_{j_m} \; I_{i_1\ldots i_n} \, I_{j_1\ldots j_m}$$

$$+ \omega^4 \, (x_\alpha x_\alpha)$$

$$- 2g^2 \sum_{n=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)n}{n!} \, ||\underline{x}||^{-(2n+4)} \, x_{i_1}\ldots x_{i_{n-1}} \, I_o$$

$$\cdot \Big(x_1 I_{1i_1\ldots i_{n-1}} + x_2 I_{2i_1\ldots i_{n-1}} + x_3 I_{3i_1\ldots i_{n-1}} \Big)$$

$$+ 2g^2 \sum_{n=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)(2n+1)}{n!} \, ||\underline{x}||^{-(2n+4)} \, x_{i_1}\ldots x_{i_n} \, I_o \, I_{i_1\ldots i_n}$$

$$- 2g\,\omega^2\,||\underline{x}||^{-3}\,(x_\alpha x_\alpha)\,I_o$$

$$- 2g^2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)n}{n!}\,\frac{1\cdot 3\ldots(2m-1)(2m+1)}{m!}$$

$$\cdot\,||\underline{x}||^{-(2n+1)}\,||\underline{x}||^{-(2m+3)}\,x_{i_1}\ldots x_{i_{n-1}}\,x_{j_1}\ldots x_{j_m}\,I_{j_1\ldots j_m}$$

$$\left(x_1 I_{1i_1\ldots i_{n-1}} + x_2 I_{2i_1\ldots i_{n-1}} + x_3 I_{3i_1\ldots i_{n-1}}\right)$$

$$+ 2\omega^2 g \sum_{n=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)n}{n!}\,||\underline{x}||^{-(2n+1)}\,x_{i_1}\ldots x_{i_{n-1}}$$

$$\cdot\,x_1 I_{1i_1\ldots i_{n-1}} + x_2 I_{2i_1\ldots i_{n-1}}$$

$$- 2\omega^2 g \sum_{n=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)(2n+1)}{n!}\,||\underline{x}||^{-(2n+3)}\,(x_\alpha x_\alpha)\,x_{i_1}\ldots x_{i_n}\,I_{i_1\ldots i_n}$$

or

$$\gamma^2 = g^2\,||\underline{x}||^{-4}\,I_o^{\,2} \tag{E4}$$

$$+ g^2\,||\underline{x}||^{-6}\,I_i\,I_i$$

$$+ 3g^2\,||\underline{x}||^{-8}\,x_i\,x_j\,I_i\,I_j$$

$$+ 6g^2\,||\underline{x}||^{-8}\,x_i\,I_j\,I_{ij}$$

$$+ 9g^2\,||\underline{x}||^{-10}\,x_i\,x_j\,I_{ki}\,I_{kj}$$

$$+ 15 \, ||\underline{x}||^{-10} \, x_i \, x_j \, x_k \, I_i \, I_{jk}$$

$$+ \frac{225}{4} \, ||\underline{x}||^{-12} \, x_i \, x_j \, x_k \, x_\ell \, I_{ij} \, I_{k\ell}$$

$$+ 4g^2 \, ||\underline{x}||^{-6} \, x_i \, I_o \, I_i$$

$$- 6g^2 \, ||\underline{x}||^{-8} \, x_i \, x_j \, I_i \, I_j$$

$$+ 9g^2 \, ||\underline{x}||^{-8} \, x_i \, x_j \, I_o \, I_{ij}$$

$$+ \omega^4 \, (x_\alpha x_\alpha)$$

$$- 2g\omega^2 \, ||\underline{x}||^{-3} \, (x_\alpha x_\alpha) \, I_o$$

$$+ 2g\omega^2 \, ||\underline{x}||^{-3} \, x_i \, I_i$$

$$+ 6g\omega^2 \, ||\underline{x}||^{-5} \, x_i \, x_\alpha \, I_{i\alpha}$$

$$- 6g\omega^2 \, ||\underline{x}||^{-5} \, x_i \, (x_\alpha x_\alpha) \, I_i$$

$$- 15g\omega^2 \, ||\underline{x}||^{-7} \, x_i \, x_j \, (x_\alpha x_\alpha) \, I_{ij}$$

$$+ o(I_{ijk})$$

up to third-order terms.

BIBLIOGRAPHY

Abramowitz, M. and Stegun, I., 1964: Handbook of Mathematical Functions, National Bureau of Standards, Washington, D.C., U.S. Government Printing Office, Washington, D.C.

Baarda, W., 1977: Measures for the accuracy of geodetic network. Proceedings International Symposium on Optimization of Design and Computation of Control Networks, Sopron, Hungary (in press).

Baarda, W., 1973: S-transformations and criterion matrices. Netherlands Geodetic Commission Publications on Geodesy, New Series (Delft, Holland) 5 (1).

Bocchio, F., 1978: Three-dimensional representations of the geopotential field and related topics. Second Course on Advanced Geodesy, Lecture Notes, Erice, University of Athens, Erice, Greece.

Bocchio, F., 1977: Tissot's ellipsoid for general potential-invariant representations of the geopotential field on the normal ellipsoidal field and applications. Boll. di Geodesia e Scienze Affini, 36, 25-32.

Bocchio, F., 1976a: On the definition of gravity anomalies in three-dimensional potential-invariant representations of the actual gravity field of the earth. Geophysical Journal Royal Astronomical Society (London, England), 44, 289-291.

Bocchio, F., 1976b: Linear and angular deformations in three-dimensional potential-invariant representations of actual gravity field of the earth and applications. Geophysical Journal Royal Astronomical Society, 46, 155-164.

Bocchio, F., 1976c: On a general potential-invariant representation of the geopotential field on the normal ellipsoidal field and applications. Boll. di Geodesia e Scienze Affini, 35, 401-411.

Bossler, J., 1973: A note on the meaning of generalized inverse solutions in geodesy. Journal Geophysical Research, 78, 2616-2617.

Bruns, H., 1878: Die Figur der Erde, Publ. Konigl. Preuss, Geod. Institute, P. Stankiewicz Buchdruckerei, Berlin, Germany.

Chovitz, B., 1956: A general formula for ellipsoid-to-ellipsoid mapping. Boll. di Geodesia e Scienze Affini, 15, 1-20.

Chovitz, B., 1954: Some applications of the classification of map projections in terms of the metric tensor of the second order. Boll. di Geodesia e Scienze Affini, 13, 47-67.

Chovitz, B., 1952: Classification of map projections in terms of the metric tensor of second order. Boll. di Geodesia e Scienze Affini, 11, 379-394.

Eeg, J. and Krarup, R., 1973: Integrated geodesy. Danish Geodetic Institute Report No. 7, Copenhagen, Denmark.

Grafarend, E., 1978a: The definition of the telluroid. Bulletin Géodésique, No. 52, 25-37.

Grafarend, E., 1978b: Dreidimensionale Geodatische Abbildungsgleichungen und die Naherungsfigur der Erde. Z. Vermessungs. and Photogram., 103, 132-140.

Grafarend, E., 1978c: Operational geodesy, In Approximation methods in geodesy, (Moritz, H. and Sunkel, H., editors), 235-284, Wichmann Verlag, Karlsruhe, Germany.

Grafarend, E., 1977: Space-time differential geodesy. Proceeding Symposium on the Changing World of Geodetic Science, Ohio State University, Columbus, Ohio, Ohio State University, Columbus, Ohio.

Grafarend, E., 1975: Three-dimensional geodesy I--The holonomity problem. Z. Vermessungs. and Photogram., 100, 269-281.

Grafarend, E., 1972: Die Freie geodätische Randwertaufgabe und das Problem der Integrationsfläche innerhalb der Intergralgleichungs-methode. Mitteilungen des Institutes für Theoretische Geodäsie der Universität Bonn, Tagung Freie Randwertaufgaben (Free Boundary Value Problems, Grafarend, E., and Weck, N., editors), Vol. 4, und Methoden und Verfahren der mathematischen Physik, ed. B. Brosowski/E. Martensen, BI-Wissenschaftsverlag, Band, Bonn, 8, 95-124.

Grafarend, E., Mueller, I., Papo, H., and Richter, B., 1979: Concepts for reference frames in geodesy and geodynamics: the reference directions. Bulletin Géodésique, No. 53, 195-213.

Grafarend, E. and Richter, B., 1978: Three-dimensional geodesy II--the datum problem. Z. Vermessungs. and Photogram., 103, 44-59.

Grafarend, E. and Schaffrin, B., 1976: Equivalence of estimable quantities and invariants in geodetic networks. Z. Vermessungswesen, 101, 485-491.

Grafarend, E. and Schaffrin, B., 1974: Unbiased free net adjustment. Survey Review 22, 200-218.

Grafarend, E. and Weck, N., 1972: Tagung "Freie Randwertaufgaben". Mitteilungen aus dem Institute fur Theoretische Geodäsie der Universität Bonn, No. 4, University FAF, Bonn, Germany.

Grafarend, E. and Niemeier, W., 1971: The free nonlinear boundary value problem of physical geodesy. Bulletin Géodésique, No. 101, 243-262.

Hirvonen, R., 1961: The reformation of geodesy. Journal Geophysical Research, 66, 1471-1478.

Hirvonen, R., 1960: New theory of gravimetric geodesy. Annual Academy of Science Fennicai A III (56), Helsinki, Finland.

Koch, K.R., 1971: Die geodätische Randwertaugfabe bei bekannter Erdoberfläche. Z. Vermessungs. and Photogram., 96, 218-224.

Koch, K.R., 1968: Solution of the geodetic boundary value problem in case of a reference ellipsoid. Department of Geodetic Science Report 104, The Ohio State University, Columbus, Ohio.

Krarup, T., 1973a: Letter III on Molodenski's problem. Internal Report of the Geodetic Institute Copenhagen, Copenhagen, Denmark.

Krarup, T., 1973b: The mathematical formulation of Molodenski's problem. Internal Report of the Geodetic Institute Copenhagen, Copenhagen, Denmark.

Krarup, T., 1969: A contribution to the mathematical foundation of physical geodesy. Geodetic Institute Report 44, Copenhagen, Denmark, 80 pp.

Lane, E.P., 1939: Metric differential geometry of curves and surfaces. University of Chicago Press, Chicago, Illinois.

Levi-Civita, R., 1926: The absolute differential calculus. Blackie, London, England.

Livieratos, E., 1978: Some topics in operational geometric differential geodesy. Second Course on Advanced Geodesy, Lecture Notes, Erice, Greece.

Livieratos, E., 1976: On the geodetic singularity problem. Manuscripta Geodaetica 1, 269-292.

Marussi, A., 1974a: On the representation of the actual gravity field of the earth on the normal ellipsoidal field. Geophysical Journal Royal Astronomical Society, 37, 347-352.

Marussi, A., 1974b: Geodetic networks in space. Canadian Surveyor, 28, 627-636

Marussi, A., 1973: On the three-dimensional computation of geodetic networks and anomalies. Rend. Acc. Naz. Lincei, Cl. Sc. Mat. Fis. Nat., Serie VIII (55), 1-7.

Marussi, A., 1952: Intrinsic geodesy. Report of program, The Ohio State Research Foundation, Project No. 485, Columbus, Ohio.

Moritz, H., 1977: Recent developments in the geodetic boundary-value problem. Department of Geodetic Science Report 266, The Ohio State University, Columbus, Ohio.

Moritz, H., 1965: The boundary value problem of physical geodesy. Annual Academy of Sciences Fennicae A III (83), Helsinki, Finland.

Nakigoglu, S.M., 1978: Variational formulation of the geodetic boundary value problem. Bulletin Géodésique, No. 52, 93-100.

Niemeier, W., 1972: Eine Formulierund der freien skalaren und vektoriellen geodatischen Randwertaufgabe. Master of Science thesis (unpublished).

O'Keefe, J.A., 1953: The isoparametric method of mapping one ellipsoid on another. EOS, Transactions American Geophysical Union 34, 869-875.

Sanso, F., 1978a: Representation of geodetic measurements and collocation in gravity space. Second Course on Advanced Geodesy, Lecture Notes, Erice, Greece.

Sanso, F., 1978b: Molodensky's problem in gravity space: A review of first results. Proceeding Symposium on Optimization of Design and Computation of Control Networks, Sopron, Hungary, (in press) and Bulletin Géodésique 52, (1978) 59-70.

Sanso, F., 1976: Discussion on the existence and uniqueness of the solution of Molodensky's problem in gravity space. Accademia Nazionale dei Lincei Serie VIII (Rome, Italy) LXI (3-4), 260-268.

Sünkel, H., 1978: Zur Geometrie des normalen Schwerefeldes. Österreichische Z. Vermessungs. and Photogram. 66, 71-85.

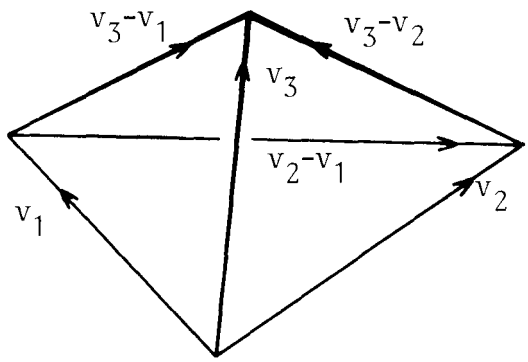


Figure 3.--Triangular network in a vector space.

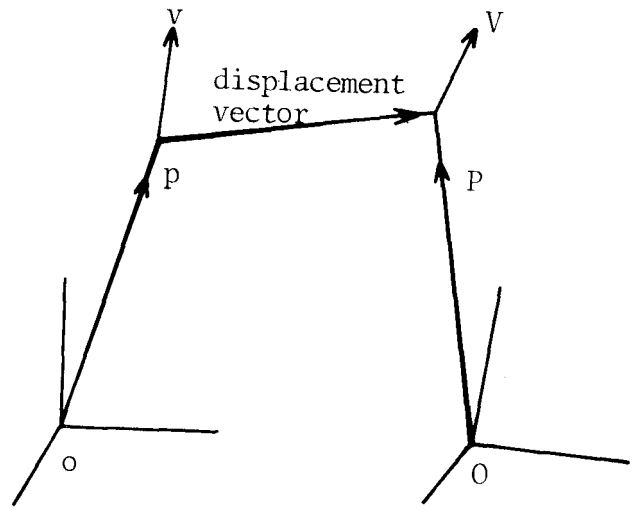


Figure 4.--The displacement vector.

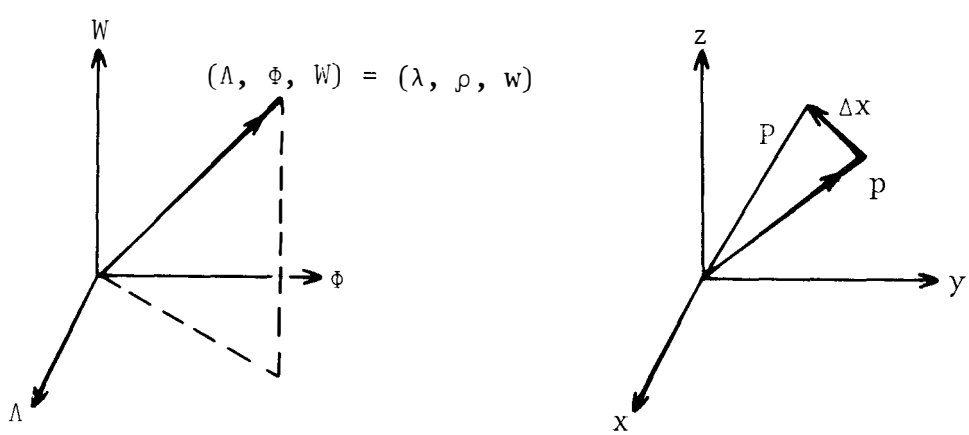


Figure 5.--Isoparametric mapping in the geometric and the gravity space for longitude, latitude, and potential.

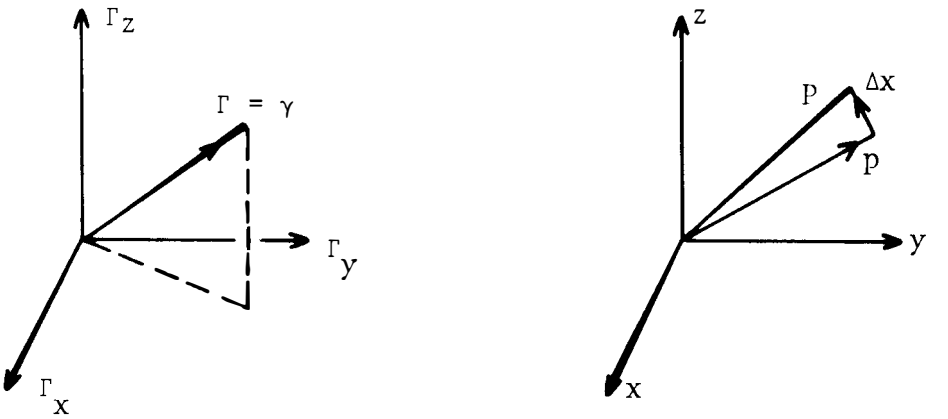


Figure 6.--Isoparametric mapping in the geometric and the gravity space for longitude, latitude, and gravity.

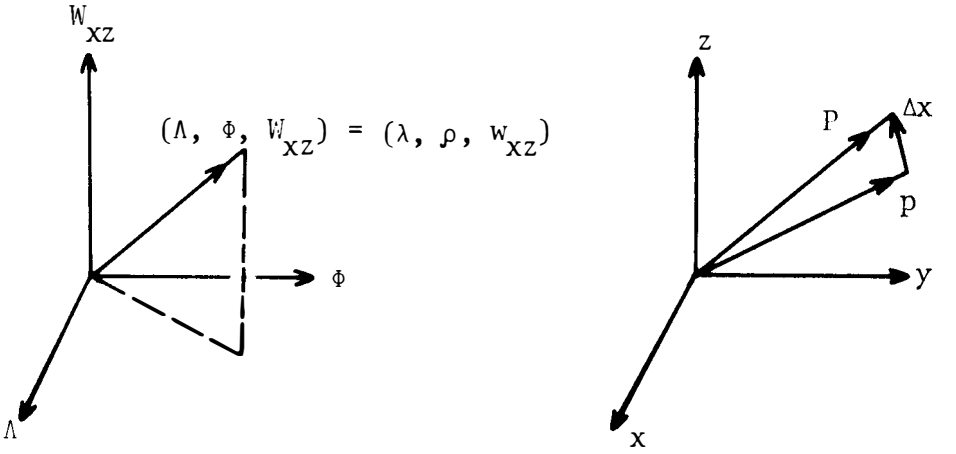


Figure 7.--Isoparametric mapping in the geometric and the gravity space for longitude, latitude, and gravity gradient.

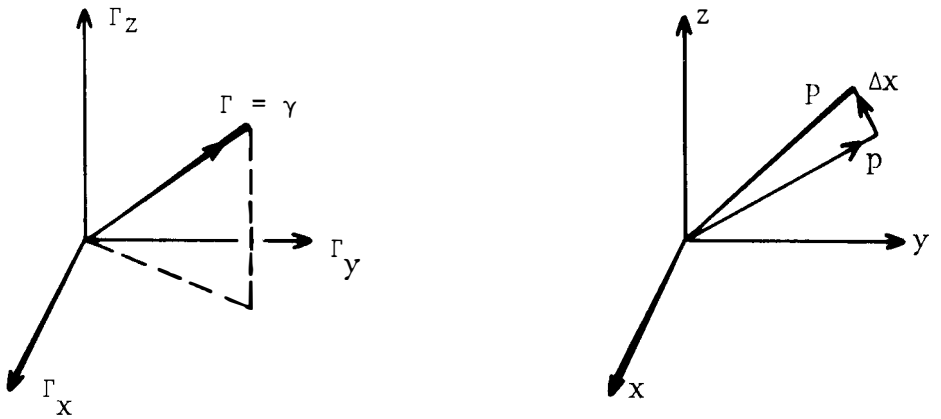


Figure 6.--Isoparametric mapping in the geometric and the gravity space for longitude, latitude, and gravity.

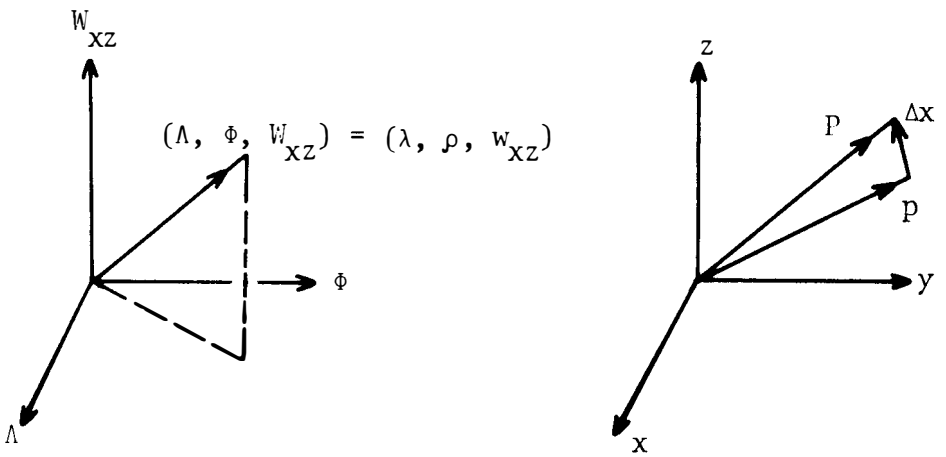


Figure 7.--Isoparametric mapping in the geometric and the gravity space for longitude, latitude, and gravity gradient.

(Continued from inside front cover)

NOS NGS-3 Adjustment of geodetic field data using a sequential method. Marvin C. Whiting and Allen J. Pope, March 1976, 11 pp (PB253967). A sequential adjustment is adopted for use by NGS field parties.

NOS NGS-4 Reducing the profile of sparse symmetric matrices. Richard A. Snay, June 1976, 24 pp (PB-258476). An algorithm for improving the profile of a sparse symmetric matrix is introduced and tested against the widely used reverse Cuthill-McKee algorithm.

NOS NGS-5 National Geodetic Survey data: availability, explanation, and application. Joseph F. Dracup, Revised January 1979, 45 pp (PB80 118615). The summary gives data and services available from NGS, accuracy of surveys, and uses of specific data.

NOS NGS-6 Determination of North American Datum 1983 coordinates of map corners. T. Vincenty, October 1976, 8 pp (PB262442). Predictions of changes in coordinates of map corners are detailed.

NOS NGS-7 Recent elevation change in Southern California. S.R. Holdahl, February 1977, 19 pp (PB265-940). Velocities of elevation change were determined from Southern Calif. leveling data for 1906-62 and 1959-76 epochs.

NOS NGS-8 Establishment of calibration base lines. Joseph F. Dracup, Charles J. Fronczek, and Raymond W. Tomlinson, August 1977, 22 pp (PB277130). Specifications are given for establishing calibration base lines.

NOS NGS-9 National Geodetic Survey publications on surveying and geodesy 1976. September 1977, 17 pp (PB275181). Compilation lists publications authored by NGS staff in 1976, source availability for out-of-print Coast and Geodetic Survey publications, and subscription information on the Geodetic Control Data Automatic Mailing List.

NOS NGS-10 Use of calibration base lines. Charles J. Fronczek, December 1977, 38 pp (PB279574). Detailed explanation allows the user to evaluate electromagnetic distance measuring instruments.

NOS NGS-11 Applicability of array algebra. Richard A. Snay, February 1978, 22 pp (PB281196). Conditions required for the transformation from matrix equations into computationally more efficient array equations are considered.

NOS NGS-12 The TRAV-10 horizontal network adjustment program. Charles R. Schwarz, April 1978, 52 pp (PB283087). The design, objectives, and specifications of the horizontal control adjustment program are presented.

NOS NGS-13 Application of three-dimensional geodesy to adjustments of horizontal networks. T. Vincenty and B. R. Bowring, June 1978, 7 pp (PB286672). A method is given for adjusting measurements in three-dimensional space without reducing them to any computational surface.

NOS NGS-14 Solvability analysis of geodetic networks using logical geometry. Richard A. Snay, October 1978, 29 pp (PB291286). No algorithm based solely on logical geometry has been found that can unerringly distinguish between solvable and unsolvable horizontal networks. For leveling networks such an algorithm is well known.

NOS NGS-15 Goldstone validation survey - phase 1. William E. Carter and James E. Pettey, November 1978, 44 pp (PB292310). Results are given for a space system validation study conducted at the Goldstone, Calif., Deep Space Communication Complex.

NOS NGS-16 Determination of North American Datum 1983 coordinates of map corners (second prediction). T. Vincenty, April 1979, 6 pp (PB297245). New predictions of changes in coordinates of of map corners are given.

NOS NGS-17 The HAVAGO three-dimensional adjustment program. T. Vincenty, May 1979, 18 pp (PB297069). The HAVAGO computer program adjusts numerous kinds of geodetic observations for high precision special surveys and ordinary surveys.

NOS NGS-18 Determination of astronomic positions for California-Nevada boundary monuments near Lake Tahoe. James E. Pettey, March, 1979, 22 pp (PB301264). Astronomic observations of the 120th meridian were made at the request of the Calif. State Lands Commission.

NOS NGS-19 HOACOS: A program for adjusting horizontal networks in three dimensions. T. Vincenty, July 1979, 18 pp (PB301351). Horizontal networks are adjusted simply and efficiently in the height-controlled spatial system without reducing observations to the ellipsoid.

NOS NGS-20 Geodetic leveling and the sea level slope along the California coast. Emery I. Balazs and Bruce C. Douglas, September 1979, 23 pp (PB80 120611). Heights of four local mean sea levels for the 1941-59 epoch in California are determined and compared from five geodetic level lines observed (leveled) between 1968-78.

NOS NGS-21 Haystack-Westford Survey. W. E. Carter, C. J. Fronczek, and J. E. Pettey, September 1979, 57 pp. A special purpose survey was conducted for VLBI test comparison.

NOS NGS-22 Gravimetric tidal loading computed from integrated Green's functions. C. C. Goad, October 1979, 15 pp. Tidal loading is computed using integrated Green's functions.

NOS NGS-23 Use of auxiliary ellipsoids in height-controlled spatial adjustments. B. R. Bowring and T. Vincenty, November 1979, 6 pp. Auxiliary ellipsoids are used in adjustments of networks in the height-controlled three-dimensional system for controlling heights and simplifying transformation of coordinates.

NOS NGS-24 Determination of the geopotential from satellite-to-satellite tracking data. B. C. Douglas, C. C. Goad, and F. F. Morrison, January 1980, 32 pp. The capability of determining the geopotential from satellite-to-satellite tracking is analyzed.

(Continued on inside back cover)

(Continued)

NOAA Technical Reports, NOS/NGS subseries

NOS 65 NGS 1 The statistics of residuals and the detection of outliers. Allen J. Pope, May 1976, 133 pp (PB258428). A criterion for rejection of bad geodetic data is derived on the basis of residuals from a simultaneous least-squares adjustment. Subroutine TAURE is included.

NOS 66 NGS 2 Effect of Geoceiver observations upon the classical triangulation network. R. E. Moose and S. W. Henriksen, June 1976, 65 pp (PB260921). The use of Geoceiver observations is investigated as a means of improving triangulation network adjustment results.

NOS 67 NGS 3 Algorithms for computing the geopotential using a simple-layer density model. Foster Morrison, March 1977, 41 pp (PB266967). Several algorithms are developed for computing with high accuracy the gravitational attraction of a simple-density layer at arbitrary altitudes. Computer program is included.

NOS 68 NGS 4 Test results of first-order class III leveling. Charles T. Whalen and Emery Balazs, November 1976, 30 pp (GPO# 003-017-00393-1) (PB265421). Specifications for releveling the National vertical control net were tested and the results published.

NOS 70 NGS 5 Selenocentric geodetic reference system. Frederick J. Doyle, Atef A. Elassal, and James R. Lucas, February 1977, 53 pp (PB266046). Reference system was established by simultaneous adjustment of 1,233 metric-camera photographs of the lunar surface from which 2,662 terrain points were positioned.

NOS 71 NGS 6 Application of digital filtering to satellite geodesy. C. C. Goad, May 1977, 73 pp (PB-270192). Variations in the orbit of GEOS-3 were analyzed for M_2 tidal harmonic coefficient values that perturb the orbits of artificial satellites and the Moon.

NOS 72 NGS 7 Systems for the determination of polar motion. Soren W. Henriksen, May 1977, 55 pp (PB274698). Methods for determining polar motion are described and their advantages and disadvantages compared.

NOS 73 NGS 8 Control leveling. Charles T. Whalen, May 1978, 23 pp (GPO# 003-017-00422-8) (PB286838). The history of the National network of geodetic control, from its origin in 1878, is presented in addition to the latest observational and computational procedures.

NOS 74 NGS 9 Survey of the McDonald Observatory radial line scheme by relative lateration techniques. William E. Carter and T. Vincenty, June 1978, 33 pp (PB287427). Results of experimental application of the "ratio method" of electromagnetic distance measurements are given for high resolution crustal deformation studies in the vicinity of the McDonald Lunar Laser Ranging and Harvard Radio Astronomy Stations.

NOS 75 NGS 10 An algorithm to compute the eigenvectors of a symmetric matrix. E. Schmid, August 1978, 5 pp (PB287923). Method describes computations for eigenvalues and eigenvectors of a symmetric matrix.

NOS 76 NGS 11 The application of multiquadric equations and point mass anomaly models to crustal movement studies. Rolland L. Hardy, November 1978, 63 pp (PB293544). Multiquadric equations, both harmonic and nonharmonic, are suitable as geometric prediction functions for surface deformation and have potentiality for usage in analysis of subsurface mass redistribution associated with crustal movements.

NOS 79 NGS 12 Optimization of horizontal control networks by nonlinear programing. Dennis G. Milbert, August 1979, 44 pp (PB80 117948). Several horizontal geodetic control networks are optimized at minimum cost while maintaining desired accuracy standards.

NOS 82 NGS 13 Feasibility study of the conjugate gradient method for solving large sparse equation sets. Lothar Gründig, February 1980, 22 pp. Method is suitable for constrained adjustments of triangulation networks but not for free adjustments.

NOS 83 NGS 14 Tidal corrections to geodetic quantities. Petr Vaníček, February 1980, 30 pp. Corrections for tidal force are formulated and tidal aspects relating to geodesy are discussed.

NOS 84 NGS 15 Application of special variance estimators to geodesy. John D. Bossler and Robert H. Hanson, February 1980. Special variance estimators, one involving the use of noninteger degrees of freedom, are analyzed and applied to least-square adjustments of geodetic control networks to determine their effectiveness.

NOAA Manuals, NOS/NGS subseries

NOS NGS 1 Geodetic bench marks. Lt. Richard P. Floyd, September 1978, 56 pp (GPO# 003-017-00442-2) (PB296427). Reference guide provides specifications for highly stable bench marks, including chapters on installation procedures, vertical instability, and site selection considerations.

U.S. DEPARTMENT OF COMMERCE
National Oceanic and Atmospheric Administration
National Ocean Survey
National Geodetic Survey, OA/C18x2
Rockville, Maryland 20852

OFFICIAL BUSINESS

POSTAGE AND FEES PAID
U.S. DEPARTMENT OF COMMERCE
COM-210
THIRD CLASS MAIL

U.S.MAIL

NOAA--S/T 80-160