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SOLVABILITY ANALYSIS OF GEODETIC NETWORKS USING LOGICAL GEOMETRY

Richard A. Snay

Rockville, Md.
October 1978

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Specifications To Support Classification, Standards of Accuracy, and General Specifications of Geodetic Control Surveys. Federal Geodetic Control Committee, John O. Phillips (Chairman), Department of Commerce, NOAA, NOS, 1975, reprinted annually 30 pp (PB261037). This publication provides the rationale behind the original publication, "Classification, Standards of Accuracy, ..." cited above.

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SOLVABILITY ANALYSIS OF GEODETIC NETWORKS USING LOGICAL GEOMETRY

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ABSTRACT. A complete solvability analysis of leveling networks can be efficiently performed on the computer without recourse to real number arithmetic. Attempts to generalize this statement to include horizontal networks have been unsuccessful. With leveling networks the solvability analysis relies on a mechanism of identifying solvable subnetworks of a solvable network. A horizontal network can be solvable and yet have no nontrivial solvable subnetworks.

INTRODUCTION

In the least-squares adjustment of a geodetic network, the question arises as to whether the observations and constraints of the network are sufficient to render the normal equations solvable. For certain small networks the question of solvability is probably best answered by a combined visual-mental process that involves the inspection of a graphical display of the network. For networks with complicated geometry or thousands of stations, this visual-mental process ranges from inefficient to impossible. Thus, an automated solution to the process is sought. One approach to automation is to use any of several different numerical techniques for determining the rank of the normal equations matrix or, equivalently, the rank of the design matrix. An alternate approach is to automate a logical process which, as with the visual-mental process above, is based solely on the geometry of the network. The second approach appears simple and pleasing in concept. However, no algorithm based solely on logical geometry has been found which can unerringly distinguish between the solvable and unsolvable horizontal networks. Such is not the case with the class of leveling networks. An efficient, universally applicable algorithm exists. In the case of leveling networks, the infinite variety of geometries generated by the various arrangements of the loops or circuits can be reduced to a finite number of repeated situations. In the horizontal case a reduction from the infinite to the finite has yet to be found, if indeed such a reduction exists. Without this reduction any algorithm based on logical geometry must necessarily involve an infinite number of logical tests and, hence, require an infinite amount of computer time. In this paper it is demonstrated why the technique that reduces the infinite variety of leveling networks to a finite set fails when applied in the horizontal case.

DEFINITIONS

A network as used here refers to a set of unknown parameters, such as the coordinates of given stations, together with a set of observations and constraints that establish mathematical relationships among the unknown parameters. Specifically, for a leveling network, each unknown parameter is the height coordinate of a station and each observation or constraint is either the numerical value for a station's height or the height difference between two stations. A horizontal network is a network whose unknown parameters consist of coordinates of given stations referred to a 2-dimensional datum surface, usually a plane, a sphere, or an ellipsoid, together with so-called orientation unknowns. The observations and constraints are either numerical values for the station coordinates or the distance, direction, or azimuth between two stations. An azimuth is the angular displacement of a line from an adopted reference orientation associated with the datum surface. A direction is the angular displacement of a line from an alternative orientation. Hence, for each set of directions that refers to a common reference orientation, there is an orientation unknown to account for the difference between this orientation and the adopted reference orientation of the datum surface.

From the viewpoint of network geometry, there is no essential difference between an observation and a constraint. Hence, the word observation is used subsequently as an all inclusive term. With both leveling and horizontal networks, observations other than those previously identified are conceivable. However, such complications are not considered here.

The term network solvability needs to be precisely defined. Optimally, the definition should correspond with the existence of a least-squares solution for the network. However, it is more convenient to define a solvable network as one for which there exists a set of initial approximations to the unknown parameters such that the normal equations' matrix is of full rank or, equivalently, the rank of the design matrix equals the number of unknown parameters. According to this definition the solvability of a network is independent of the values assumed by the observations. For nonlinear least-squares models, such as is the case for horizontal networks, the existence of a least-squares solution is dependent on the observational values. The following example illustrates the discrepancies that can result between the two concepts because of this.

Let A, B, and C denote three horizontal stations in a plane such that the coordinates of stations A and B are observed together with the azimuths from A to C and from B to C. Three basic geometric situations can occur depending on the values assigned to the two azimuths (see fig. 1). In the first case (fig. 1a) the azimuth lines intersect at a single point. Here the least-squares solution exists. In the second case (fig. 1b) the azimuth lines do not intersect. Here the least-squares solution does not exist because there are no positions for stations A, B, and C at which the weighted sum of squares of the residuals attains a minimum. (This excludes positions for which stations are collocated.) In the third case (fig. 1c) the azimuth lines intersect at infinitely many points. Here, since the least-squares condition is satisfied for infinitely many positions of C, the least-squares solution is not considered to exist. In all three cases there are initial positions for which the rank of the design matrix equals six. Hence, each of these networks is solvable with respect to the definition.

In the second case of the example, if it is assumed that the two azimuths were physically observed, then one of the values assigned to these azimuths must have been in error. Without this error the least-squares solution would have existed. This example illustrates that algorithms to test for solvability by the definition are incapable of detecting these errors in the data. The third case illustrates that such algorithms are also incapable of detecting networks of critical configuration. Critical configurations in satellite networks have been studied by Rinner (1966), Blaha (1971), and Tsimis (1972 and 1973). No similar study for horizontal networks is known to the author. Critical configurations do not occur with leveling networks since they belong to the class of linear least-squares models.

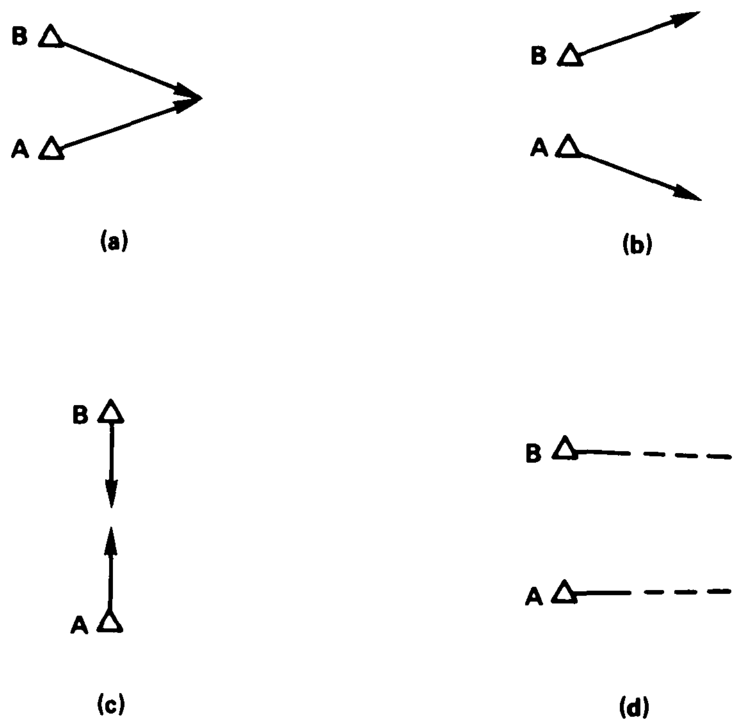


Figure 1.--Network geometry depends on observational values.

The term ill-conditioning refers to another problem related to solvability. For an ill-conditioned system the least-squares solution exists but is computationally difficult to obtain. Figure 1d illustrates such a situation. Geometrically, the network of figure 1d is equivalent to the first case in that the azimuth lines intersect at a unique point. However, the two lines are so nearly parallel that the solution might not be obtainable by a standard solution algorithm as it is coded within a computer. By definition this network is solvable. Again the discrepancy results because solvability is defined to be independent of observational values.

CRITERIA FOR THE EVALUATION OF A SOLVABILITY CHECK

Because (1) numerical techniques for determining the rank of a matrix, such as Gram-Schmidt orthogonalization, involve essentially as much work as is required in obtaining the least-squares solution, and (2) the solution is actually what is desired, the solution might be attempted without a solvability check. In this way if the network is solvable, then there is no wasted effort; and if the network is not solvable, then the mathematical algorithm used to obtain the solution usually ascertains this. There are two shortcomings with this approach to the problem of solvability.

For the purpose of illustration, assume that the least-squares solution is attempted using a direct or elimination technique, such as Gauss-Doolittle, for reducing the matrix of normal equations. If the network is not solvable, then the solution algorithm will detect this when an element on the main diagonal of the matrix reduces to zero. Herein lies the first shortcoming, namely, the inability of the computer to recognize zero. As people and computers can only manipulate a finite number of significant figures, it is probable that the particular main diagonal element of the unsolvable network will not reduce exactly to zero. To compensate for this a small positive number r can be chosen as a tolerance, in which case the number x is said to equal zero whenever $|x| < r$. However, with any given tolerance, it is possible either to call a nonzero number zero or to have a number greater than r which is theoretically zero. In the first case a solvable network is called unsolvable. In the second case an unsolvable network is called solvable.

The second shortcoming of the method occurs when the network has been determined to be unsolvable and it becomes desirable to know which combinations of observations (and/or constraints) can be added into the network to insure solvability. While it is true that the location of the diagonal element which becomes zero during the reduction contains some information of this nature, more information about the geometry of the network and the observations is required for this type of analysis.

In view of the preceding, the following criteria can be used to evaluate a proposed solvability checking procedure:

1. The procedure should require significantly less resources than the process of obtaining the least-squares solution.
2. The procedure should be foolproof in distinguishing between solvable and unsolvable networks. In particular, the procedure should not depend on the recognition of calculated zeroes.

3. The procedure should provide sufficient information concerning the geometry of the network for effective analysis of unsolvable networks.

LOGICAL GEOMETRY

A complete solvability check based on logical geometry is said to exist for a class of networks if there is a finite set of conditions which can be used to distinguish between solvable and unsolvable networks. Moreover, it is required that each such condition be testable without the use of real number arithmetic. To illustrate the concept, consider the class of horizontal networks where each member consists of three stations in the plane. Furthermore, the observations for members of this class include the two coordinates for one station and an azimuth between two stations. All other observations are either distances or angles (the clockwise difference between two directions which refer to a common initial orientation). For networks of this class a complete solvability check based on logical geometry can be formulated with three conditions. Specifically, a network in this class is solvable if and only if the observations include one of the following combinations:

1. three distances,
2. two distances and one angle, or
3. one distance and two angles.

The distinction between a complete and an incomplete solvability check is of significance. The difference is based on whether or not the solvability check is infallible in distinguishing between solvable and unsolvable networks. An example of an incomplete solvability check for horizontal networks is the condition that each station of the network be involved in at least two observations. The incompleteness of this solvability check is demonstrated by a horizontal network with observations for the two coordinates of one station and distances between each possible pair of stations. Although each station has two observations, the network has no orientation control and is unsolvable. In the absence of a complete solvability check for a class of networks, the incomplete checks have found some application. The particular one mentioned above satisfies two of the criteria established in the previous section: it is computationally efficient since counting is the only numerical operation involved and it provides usable information about the geometry of the network.

THE INDUCTION PRINCIPLE

Consider the mathematical statement: if 2^n-1 is a prime number for an integer n greater than 1, then $2^{n-1}(2^n-1)$ is the sum of all its proper divisors. (A proper divisor of a number is a factor of the number other than the number itself.) For example, if $n = 3$, then $2^n - 1 = 7$ is a prime number. Thus, $2^{n-1}(2^n-1) = 28$ is the sum of all its proper divisors or $28 = 1 + 2 + 4 + 7 + 14$. The mathematical statement actually involves an infinite number of statements, one for each integer greater than one. Hence, its validity cannot be established by checking the truth of the statement for all values of n up to 100 or for all values up to 100,000,000. To validate such a statement a technique called mathematical induction is usually appropriate, although it can often be circumvented. The basic step in using induction is to demonstrate that the validity of the statement for the case of n is dependent upon the validity of the statement for the cases of certain numbers less than n.

In the problem of solvability, one encounters the situation of having an infinite number of different possible geometries since a network can have any positive number of stations. Thus, one approach to the solvability problem is to apply the induction principle. The crucial step in the application involves the formulation of the induction hypothesis, i.e., finding the proper condition which associates the geometry of a network to the geometry of a network with fewer stations.

THE INDUCTION PRINCIPLE APPLIED TO LEVELING NETWORKS

For leveling networks the induction principle can be applied to develop a complete solvability check based on logical geometry. The formulation of the induction hypothesis is based on the validity of theorem 1.

Theorem 1. A leveling network is solvable if and only if it contains a station with at least one associated observation such that the network obtained by removing this station and its associated observation(s) is solvable.

From theorem 1 it follows that every solvable network can be constructed one station at a time with each intermediate network being solvable. This leads to a readily testable condition for solvability as stated in theorem 2 below. The statement of theorem 2 makes reference to the correspondence between the set of leveling networks and the set of undirected graphs. In this correspondence each station or height unknown relates to a vertex of a graph and each observed height difference relates to an edge of this graph. A component of a graph G is a connected subgraph of G which is not contained in any larger connected subgraph of G. A subgraph H is connected if each pair of vertexes of H can be linked by a sequence of edges in H. It follows that each vertex of a graph belongs to one and only one component.

Theorem 2. A leveling network is solvable if and only if each component of the associated graph contains a vertex which corresponds to a station whose height has been observed.

The proofs of theorems 1 and 2 are given in appendix A. The application of theorem 2 is easily automated. From the observations a table of the existing edges of the graph is constructed and the components are identified. After this, it is only a matter of checking whether or not each component has at least one observed height.

INAPPLICABILITY OF INDUCTION PRINCIPLE FOR HORIZONTAL NETWORKS

The induction principle, although successful in developing a complete solvability check for the class of leveling networks, has yet to be applied to the horizontal case. An effective induction hypothesis has yet to be formulated that relates the geometry of a horizontal network to the geometry of horizontal networks with fewer stations or fewer observations. In the leveling case the induction hypothesis is based on the property that a solvable network can be built up one station at a time with each intermediate network being solvable. Such is not the case with solvable horizontal networks. There are solvable horizontal networks which cannot be built from a simple network one station at a time with each intermediate network being solvable. Moreover, for each positive integer n, there is a solvable network which cannot be built from a simple network with less than n stations at a time and each intermediate network being solvable. This statement follows by showing that for each positive integer k, there is a solvable horizontal network with more than k stations which does not contain any smaller solvable subnetworks of more than two stations. To simplify the arguments in this proof it is convenient to deal with networks with only one type of observation. Thus, a trilateration network is defined as a horizontal network where the only unknown parameters are station coordinates and the only observations are distances between pairs of stations. A trilateration network is said to be rigid if the normal equations' matrix has rank 2n - 3 where n is the number of stations. Hence, a rigid trilateration network can become a solvable horizontal network with the addition of certain combinations of three observations of position and/or azimuth. For leveling networks the concept of rigidity applies to networks where the only observations are height differences. Here a network is rigid if the rank of the matrix is n - 1, where n is the number of unknown heights parameters. It follows that a leveling network is rigid if and only if the associated graph has only one component. Examples of rigid trilateration networks are pictured in figure 2. Figure 3 illustrates some nonrigid trilateration networks.

Let T denote the simplest rigid trilateration network, i.e., two stations and a distance as pictured in figure 2a. Figures 2b and 2c are examples of rigid trilateration networks which can be constructed from T by adding one station at a time with each intermediate network being rigid. Figure 2d illustrates a rigid trilateration network which cannot be so constructed because it has no rigid subnetworks of three stations (no triangles). In view of this, one might ask whether or not there exists some finite variety of "building blocks" such that every rigid trilateration network can be constructed from T by adding one block at a time with each intermediate network being rigid. This principle is suggested by the network of figure 2e. Unfortunately, such is not the case. The variety of building blocks is

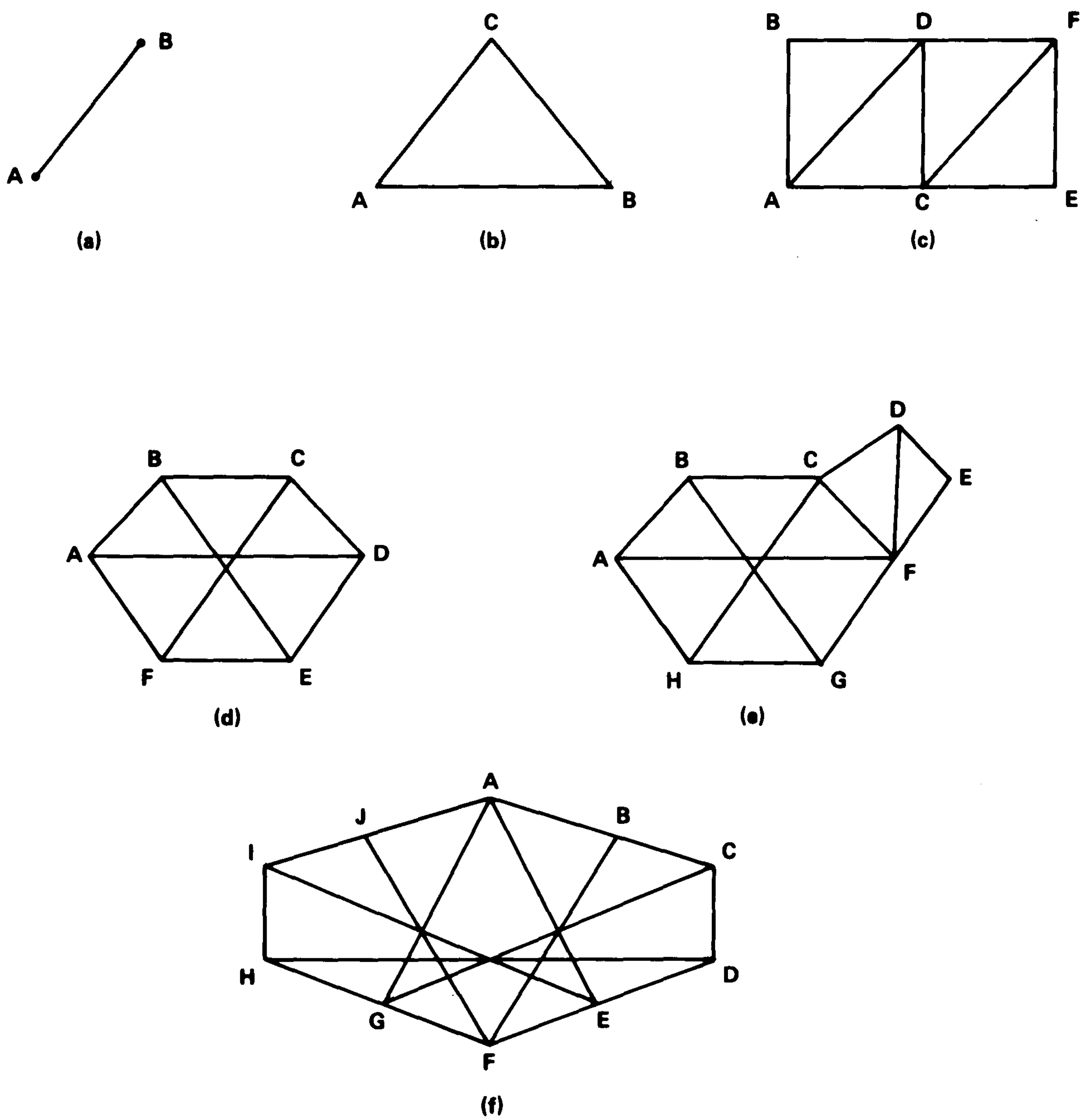


Figure 2.--Rigid trilateration networks.

infinite. To prove this it is shown that, for each integer k, there is a rigid trilateration network with more than k stations which contains no rigid subnetworks other than T. Thus, an infinite sequence of rigid trilateration networks is identified. Figure 4 illustrates the first three members of this set, the set of "staircase networks" as suggested by their geometry. Specifically, for each integer n greater than 2, the staircase network N_{2n} consists of 2n stations and 4n - 3 distance observations. Let $d(P_j, P_k)$ represent the distance between stations P_j and P_k; then the staircase network N_{2n} for $n \geq 3$ contains the following observations:

$$d(P_i, P_{i+1}) \text{ for } 1 \leq i \leq 2n - 1$$

$$d(P_i, P_{i+3}) \text{ for } 1 \leq i \leq 2n - 3$$

$$d(P_1, P_{2n}) \ .$$

A proof that the network N_{2n} is rigid and contains no rigid subnetwork other than itself and the simple two station subnetworks is given in appendix B. Thus, an induction hypothesis based on the combination of rigid networks cannot be formulated.

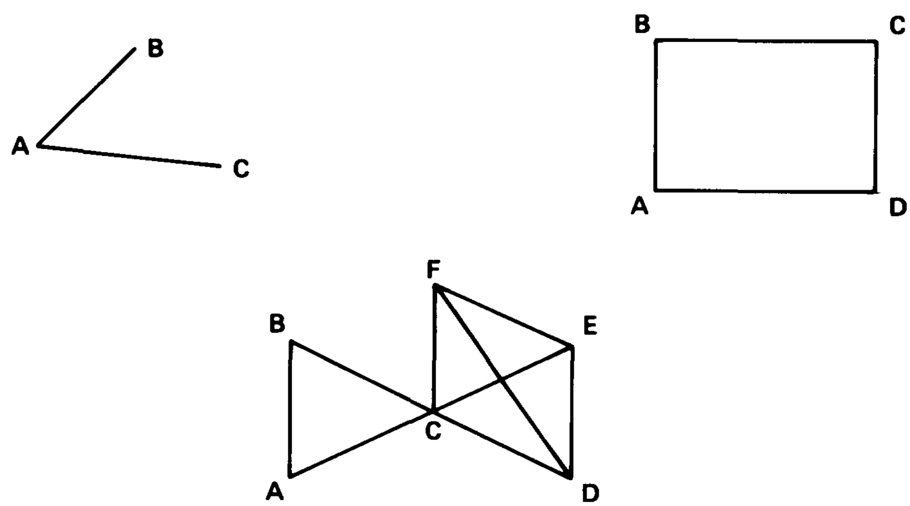
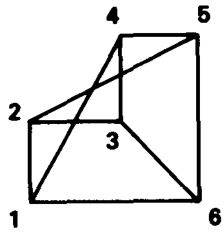
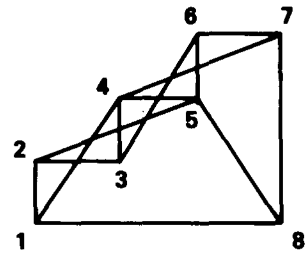


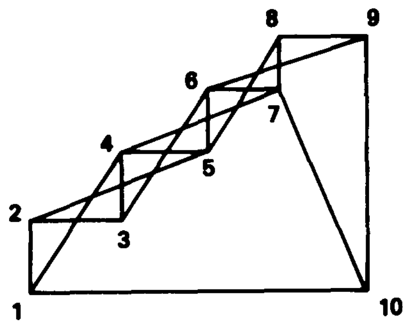
Figure 3.--Nonrigid trilateration networks.



N_6



N_8



N_{10}

Figure 4.--The first three staircase networks.

DISCUSSION

The arguments of this paper do not prove that complete solvability check based on logical geometry does not exist for horizontal networks. They do demonstrate that the technique applicable for leveling networks cannot be directly generalized to work in the horizontal case. To handle the infinite number of geometric possibilities, an alternate induction hypothesis might be formulated. The induction hypothesis presented and discredited attempted to extend quasi-solvability, i.e., rigidity, from small networks to larger networks using a finite variety of steps. Possibly an induction hypothesis can be formulated based on extending some other property, or possibly the induction principle can be ignored all together. The approach that seems the most plausible is to translate the problem into the language of graph theory and/or finite groups and look for a property like connectivity which proved fruitful in the leveling case.

ACKNOWLEDGMENT

The author is indebted to Allen Pope of the National Geodetic Survey for many interesting discussions on the topic of solvability.

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APPENDIX A.--PRESENTATION OF THEOREMS RELATED TO LEVELING NETWORKS

The proofs of theorems 1 and 2 are presented here. A tree, as referred to below, is a connected graph which becomes disconnected upon the removal of any of its edges. Equivalently, a tree is a connected graph with no loops, e.g., the graph of figure 5. It follows that a tree of n vertexes contains n - 1 edges.

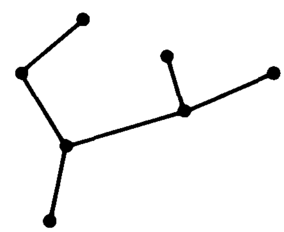

Figure 5.--A tree is a graph with no loops.

Lemma A. Let L be a leveling network of n stations whose only observations are height differences and let the graph associated with L be a tree; then the design matrix for L has rank n - 1.

Proof:
The proof is by induction on the number of stations in L. The lemma is clearly true if L has one station. Assuming the lemma's validity for the case of n stations, suppose L has n + 1 stations. Remove a vertex joined by exactly one edge. The resulting network with n stations still has a tree for the associated graph, and, by induction, its design matrix has rank n - 1. Augmenting this design matrix with removed unknown and observation results in a design matrix of rank n.

Lemma B. If L is a solvable leveling network whose associated graph is connected, then L contains a station whose height is observed.

Proof:

Suppose L has n stations; then, since the associated graph of L is connected, the graph contains a tree of n - 1 edges. By lemma A, the edges of this tree correspond to n - 1 linearly independent observed height differences. Given any observed height difference in L, it can be written as a linear combination of the observations corresponding to edges of the tree. In particular let <u, w> be an edge corresponding to an observed height difference in L; then, there are vertexes in the graph $v_1, v_2, \ldots, v_m$ such that $\langle v_i, v_{i+1}\rangle$ is an edge of the tree for $1 \leq i \leq m-1$, $v_1 = u$, $v_m = w$, and the row in the design matrix corresponding to <u,w> is a linear combination of the m-1 rows corresponding to the edges $\langle v_i, v_{i+1}\rangle$ (fig. 6). Hence, the observations corresponding to height differences account for only n-1 linearly independent rows in the design matrix. Since the design matrix has rank n, there must be an observed height in L.

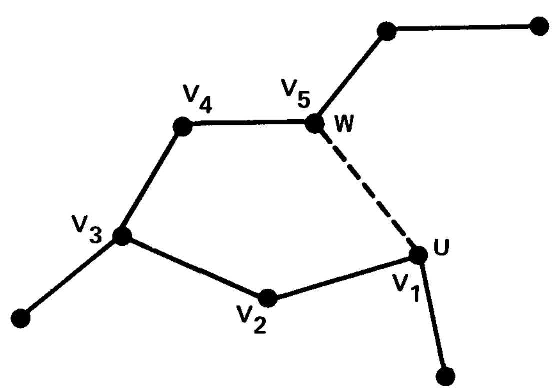

Figure 6.--In a graph of a leveling network a loop corresponds to the existence of a linearly dependent observation.

<u>Theorem 1</u>. For leveling networks the following are equivalent:

a) The network is solvable.

b) The network contains one station with at least one associated observation such that the network obtained by removing this station and its associated observation(s) is solvable.

Proof:

(a => b): Pick a component of the associated graph. Without loss of generality, it can be assumed that the station unknowns are ordered such that the design matrix A can be partitioned as

$$A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

where A_1 is the matrix whose columns correspond to the unknown heights of the stations in the chosen component. Since there are no observations between different components, no row of A contains nonzero elements in both members of the partition. Thus, as the rank of A equals the number of stations in the network, the rank of A_1 must equal the number of stations in the chosen component, say n. By lemma B, one of the rows in A_1 must correspond to an observed height. Since the rank of A_1 is n, there must be n - 1 other rows in A_1 such that these n rows are linearly independent. Let A_1* be the submatrix of A_1 corresponding to these n rows. The matrix A_1* contains at most 2n-1 nonzero elements. Hence, there exists a column in A_1* which has exactly one nonzero element. Eliminating this column and the row containing this nonzero element from A_1* leaves a matrix of rank n - 1. Hence, the network obtained by removing the corresponding station and its associated observations is solvable.

(b => a): Let L be a network containing a station with at least one associated observation such that the network obtained by removing this station and its associated observations from L has a design matrix A of rank n - 1 where n equals the number of stations in L. Augment A by an additional column for the removed station and additional rows for the removed observations. Since only the new observations have nonzero elements in the new column, a design matrix of rank n is obtained. Hence, L is solvable.

Theorem 2. For leveling networks the following are equivalent.

a) The network is solvable.

b) Each component of the graph associated with the network contains a vertex corresponding to a station whose height has been observed.

Proof:

(a => b): The proof is by induction on the number of stations in the network L. The statement is clearly true if L has only one station. By theorem 1, L contains a station with at least one associated observation such that the network L* obtained by removing this station and its associated observations is solvable. By the induction hypothesis each component of the associated graph of L* contains a vertex which corresponds to an observed height. The graph corresponding to the network L has no additional components unless the removed station has only an observed height. In either case, each component of the graph associated with L has an observed height.

(b => a): The proof is by induction on the number of stations in the network. If the network has only one station, then the statement is clearly true. Suppose the network has several stations. Choosing one of the components, a vertex can be removed from it such that each of the remaining components has an observed height. This is clearly possible if the component has only one vertex. If the component has more than one vertex, a vertex can be removed because every connected graph contains a tree with at least two vertexes and the removal of these does not disconnect the tree or the graph. One of these two vertexes can be removed such that the remainder of the component contains an observed height. Hence, the network corresponding to the graph with the removed vertex is solvable by application of the induction hypothesis and the original network is solvable by application of theorem 1.

APPENDIX B.--PRESENTATION OF THEOREMS RELATED TO THE STAIRCASE NETWORK

Here it is demonstrated that for each integer greater than 2 the staircase network N_{2n} is rigid; but no subnetwork of N_{2n} other than the trivial ones (two stations with included distance) is rigid. Recall that the staircase network contains 2n stations denoted $P_1, P_2, \ldots, P_{2n}$ and $4n - 3$ distance observations:

$$d(P_i, P_{i+1}) \text{ for } 1 \leq i \leq 2n-1$$

$$d(P_i, P_{i+3}) \text{ for } 1 \leq i \leq 2n-3$$

$$d(P_1, P_{2n}).$$

The following preliminary positions are assumed for the unknown coordinates (x_k, y_k) of station P_k for $1 \leq k \leq 2n$:

$$\left(x_k, y_k\right) = \begin{cases} \left(\frac{k-1}{2}, \frac{k-1}{2}\right) & \text{if } k \text{ is odd} \\ \left(\frac{k-2}{2}, \frac{k-2}{2} + 1\right) & \text{if } k \text{ is even and } k \neq 2n \\ (n-1, 0) & \text{if } k = 2n. \end{cases}$$

<u>Theorem 3</u>. In a plane the staircase network N_{2n} is rigid for $n \geq 3$.

Proof:

In a plane the observation equation fcr an observed distance s between two stations whose coordinates are (x_1, y_1) and (x_2, y_2) is

$$s + v = \left((x_1-x_2)^2 + (y_1-y_2)^2\right)^{\frac{1}{2}}.$$

Thus, the linearized observation equation becomes

$$(s-s_0) + v = \frac{(\xi_1 - \xi_2)}{s_0} dx_1 - \frac{(\xi_1 - \xi_2)}{s_0} dx_2 + \frac{(\eta_1 - \eta_2)}{s_0} dy_1 - \frac{(\eta_1 - \eta_2)}{s_0} dy_2$$

where $s_0 = \left((\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2\right)^{1/2}$ and (ξ_1, η_1) and (ξ_2, η_2) are approximations to the adjusted coordinates of the stations. With the observations in the order

$d(1, 2)$, $d(1, 4)$, $d(2, 3)$, $d(2, 5)$, ..., $d(i, i+1)$, $d(i, i+3)$, ...,
$d(2n-3, 2n-2)$, $d(2n-3, 2n)$, $d(2n-2, 2n-1)$, $d(2n-1, 2n)$, $d(1, 2n)$

the design matrix A_{2n} has the form illustrated in figure 7.

The rigidity of the staircase network is demonstrated by showing that the design matrix A_{2n} has rank $4n - 3$. The rank of A_{2n} equals the rank of the matrix B_{2n} where B_{2n} is obtained from A_{2n} with the following steps for $G = (1/5)^{1/2}$ and $H = 1/(n^2 - 4n + 5)^{1/2}$:

1) For $0 \le k \le n-3$, replace row $4k+2$ of A_{2n} by $1/G$ times row $4k+2$ minus 2 times row $4k+1$ plus row $4k+7$,
2) For $0 \le k \le n-4$, replace row $4k+4$ of A_{2n} by $1/G$ times row $4k+4$ minus 2 times row $4k+3$ plus row $4k+9$,
3) For $k=n-3$, replace row $4k+4$ of A_{2n} by $1/G$ times row $4k+4$ minus 2 times row $4k+3$ minus row $4k+8$,
4) For $k=n-2$, replace row $4k+2$ of A_{2n} by $1/H$ times row $4k+2$ minus row $4k+1$.

The matrix B_{2n} is illustrated in figure 8.

Consider the vectors $\bar{v}_i$ for $1 \le i \le 4n - 3$ formed from the $4n-3$ rows of the matrix B_{2n}. To demonstrate that the rank of B_{2n} is $4n - 3$ is equivalent to showing that if

ROW	P(1)	P(2)	P(3)	P(4)	P(5)	P(6)	P(7)	P(2N-5)	P(2N-4)	P(2N-3)	P(2N-2)	P(2N-1)	P(2N)
1	0 -1	0 1											
2	-G -2G			G 2G									
3		-1 0	1 0										
4		-2G -G			2G G								
5			0 -1	0 1									
6			-G -2G			G 2G							
7				-1 0	1 0								
8				-2G -G			2G G						
4N-11								0 -1	0 1				
4N-10								-G -2G			G 2G		
4N-9									-1 0	1 0			
4N-8									-2G -G			2G G	
4N-7										0 -1	0 1		
4N-6										(N-2)H -H			-(N-2)H H
4N-5											-1 0	1 0	
4N-4												0 1	0 -1
4N-3	-1 0												1 0

WHERE G = (1/5)**0.5
AND H = (1/(N**2 - 4N + 5))**0.5

Figure 7.--The matrix A_{2n}.

ROW	P(1)	P(2)	P(3)	P(4)	P(5)	P(6)	P(7)	P(2N-5)	P(2N-4)	P(2N-3)	P(2N-2)	P(2N-1)	P(2N)
1	0 -1	0 1											
2	-1 0	0 -2		0 2	1 0								
3		-1 0	1 0										
4		0 -1	-2 0		2 0	0 1							
5			0 -1	0 1									
6			-1 0	0 -2		0 2	1 0						
7				-1 0	1 0								
8				0 -1	-2 0		2 0						
4N-11								0 -1	0 1				
4N-10								-1 0	0 -2		0 2	1 0	
4N-9									-1 0	1 0			
4N-8									0 -1	-2 0		2 0	0 1
4N-7										0 -1	0 1		
4N-6										N-2 0	0 -1		-(N-2) 1
4N-5											-1 0	1 0	
4N-4												0 1	0 -1
4N-3	-1 0												1 0

Figure 8.--The matrix B_{2n}.

b_i is a set of real numbers with $\sum_{i=1}^{4n-3} b_i \overline{v}_i = \overline{0}$.

Then $b_i = 0$ for $1 \leq i \leq 4n - 3$, where $\overline{0}$ denotes the zero vector.

For $0 \leq k \leq n-2$, $b_{4k+1} = 0$ because row 4k+1 has the only nonzero element in column 4k+2.

For $0 \leq k \leq n-2$, $b_{4k+3} = 0$ because row 4k+3 has the only nonzero element in column 4k+3. Also, $b_{4n-4} = 0$ because row 4n-4 has the only nonzero element in column 4n-2.

Let C_{2n} be the matrix obtained from B_{2n} by deleting

a) row 4k + 1 and column 4k + 2 for $0 \leq k \leq n - 2$,

b) row 4k + 3 and column 4k + 3 for $0 \leq k \leq n - 2$, and

c) row 4n - 4 and column 4n - 2.

The matrix C_{2n} is illustrated in figure 9.

$$\text{Let} \quad a_j = \begin{cases} b_{2j}, & \text{if } 1 \leq j \leq 2n-3 \\ \\ b_{4n-3}, & \text{if } j = 2n-2 . \end{cases}$$

Then, since $\sum_{i=0}^{4n-3} b_i \overline{v}_i = \overline{0}$, it follows from above that

$$\sum_{j=0}^{2n-2} a_j \overline{w}_j = \overline{0} \qquad (1)$$

where $\overline{w}_j$ are the rows of the matrix C_{2n} for $1 \leq j \leq (2n-2)$.

ROW	P(1)	P(2)	P(3)	P(4)	P(5)	P(6)	P(7)	P(2N-5)	P(2N-4)	P(2N-3)	P(2N-2)	P(2N-1)	P(2N)	
1	-1	-2		2	1									
2		-1	-2		2	1								
3			-1	-2		2	1							
4				-1	-2		2							
2N-5								-1	-2		2	1		
2N-4									-1	-2		2	0	1
2N-3										N-2	-1		-N+2	1
2N-2	-1												1	0

Figure 9.--The matrix C_{2n}.

Inspecting the columns of C_{2n}, the vector eq. (1) is seen as being equivalent to the following set of equations.

Column	Equation
1	$a_{2n-2} = -a_1$
2	$a_2 = -2a_1$
3	$a_3 = -2a_2$
4	$a_4 = -2a_3+2a_1$
k (for $5 \leq k \leq 2n-4$)	$a_k = -2a_{k-1} + 2a_{k-3}+a_{k-4}$
2n-1	$2a_{2n-4}+a_{2n-5} = 0$
2n	$(n-2)\ a_{2n-3} = a_{2n-2}$

If $a_1=0$, then it follows from these equations that $a_j=0$ for $1 \leq j \leq 2n-2$. This implies $b_i=0$ for $1 \leq i \leq 4n-3$ and that the matrix B_{2n} has rank 4n-3. But if $a_1 \neq 0$, then a contradiction arises in the sense that not all of the above relationships can hold. Suppose $a_1 \neq 0$; without loss of generality, it can be assumed that $a_1=1$ since $\sum a_j\bar{w}_j = 0$ implies $\sum (ca_j)\bar{w}_j = \bar{0}$ for any constant number c. Hence the sequence $\left\{a_k\right\}_{k=1}^{2n-4}$ must satisfy the recursive relations,

$$a_1 = 1$$

$$a_2 = -2$$

$$a_3 = 4$$

$$a_4 = -6$$

$$a_k = -2a_{k-1}+2a_{k-3}+a_{k-4} \quad \text{for } 5 \leq k \leq 2n-4.$$

Thus, by lemma C stated below, $\left|a_{2n-4}\right| > \left|a_{2n-5}\right|$.

This contradicts the equation $2a_{2n-4} + a_{2n-5} = 0$ from column 2n - 1.

Hence, $a_1 = 0$ and the matrices B_{2n} and A_{2n} have rank $4n - 3$ and the staircase network is rigid.

Lemma C. The sequence of real numbers $\left\{a_k\right\}_{k=1}^{\infty}$, defined by the recursive relation

$$a_k = -2a_{k-1} + 2a_{k-3} + a_{k-4} \quad \text{for } k \geq 5 \tag{2}$$

and the initial conditions $a_1 = 1$, $a_2 = -2$, $a_3 = 4$, $a_4 = -6$, has the property that $\left|a_k\right| \geq \left|a_{k-1}\right|$ for $k \geq 2$.

Proof:

First it is shown that the sequence satisfies the equation,

$$a_k = \begin{cases} \left(\frac{k+1}{2}\right)^2 & \text{if } k \text{ is odd} \\ -\left[\left(\frac{k}{2}\right)^2 + \frac{k}{2}\right] & \text{if } k \text{ is even}. \end{cases} \tag{3}$$

The proof is by induction on k. The validity of eq. (3) is verified directly for a_1, a_2, a_3, and a_4. Assume k is an odd integer greater than 4; then, employing the induction hypothesis to substitute from eq. (3) into (2) yields

$$a_k = +2\left[\left(\frac{k-1}{2}\right)^2 + \left(\frac{k-1}{2}\right)\right] -2\left[\left(\frac{k-3}{2}\right)^2 + \left(\frac{k-3}{2}\right)\right] + \left[\frac{k-3}{2}\right]^2$$

which upon simplification becomes $a_k = \left(\frac{k+1}{2}\right)^2$.

If k is an even integer greater than 4, then substitution of (3) into (2) yields

$$a_k = -2\left(\frac{k}{2}\right)^2 + 2\left(\frac{k-2}{2}\right)^2 - \left[\left(\frac{k-4}{2}\right)^2 + \left(\frac{k-4}{2}\right)\right]$$

which upon simplification becomes $a_k = -\left[\left(\frac{k}{2}\right)^2 + \frac{k}{2}\right]$.

Hence eq. (3) holds for all positive integral values of k.

Consider the inequality $\left|a_k\right| \geq \left|a_{k-1}\right|$. (4)

If k is even, then (4) becomes $\left(\frac{k}{2}\right)^2 + \frac{k}{2} \geq \left(\frac{k}{2}\right)^2$. (5)

If k is odd, then (4) becomes $\left(\frac{k+1}{2}\right)^2 \geq \left(\frac{k-1}{2}\right)^2 + \left(\frac{k-1}{2}\right)$. (6)

The validity of inequalities (5) and (6) can be verified by straightforward algebraic manipulation. This completes the proof of the lemma.

<u>Theorem 4</u>. For $n \geq 3$, the staircase network N_{2n} has no rigid subnetworks other than the entire network and the subnetworks consisting of just two stations and one observation.

Proof:

It will be shown that every set of k stations in the staircase network N_{2n} contains at most k - 4 observations when $2 < k < 2n$. The proof is by induction on n (half the number of stations in N_{2n}). For n=3, the validity of the statement for N_6 (fig. 4) is verified by inspection, considering all possible subnetworks which contain 3, 4, or 5 stations. Suppose n>3, A is a subnetwork of N_{2n} with k stations such that 2<k<2n, and B is the subnetwork of A that is contained in N_{2n}*. (N_{2n}* is defined as the subnetwork of N_{2n} containing all stations except P_{2n-2} and P_{2n-1} (fig. 10). Let j equal the number of stations in B. Three separate cases are considered:

Case 1: j = 1, or 2. Here k is 3 or 4 and A contains stations P_{2n-2} or P_{2n-1} or both. The case of k=3 is eliminated since N_{2n} contains no triangle involving P_{2n-2} or P_{2n-1}. If k=4, then P_{2n-2}, and P_{2n-1} must both belong to A. Inspection of the network reveals that there are no four stations in N_{2n} including both P_{2n-2} and P_{2n-1} and having more than four distance observations between pairs of them. Thus A has at most 2k-4 observations.

Case 2: $2<j<2(n-1)$. Since N_{2n}* is a subnetwork of $N_{2(n-1)}$, it follows by the induction hypothesis that B contains 2j-4 or fewer observations. If A=B, i.e., j=k, then A has at most 2k-4 observations. Now if A is obtained from B by adding just P_{2n-2} or P_{2n-1} but not both, i.e., k=j+1, then A has at most only two more observations than B. Hence, A has 2k-4 or fewer observations. If A is obtained from B by adding both P_{2n-2} and P_{2n-1}, i.e., k=j+2, then A can have as many as five more observations than B (the dashed lines of fig. 10). However, if A does have five more observations than B, then both stations P_{2n-5} and P_{2n} are in B, in which case B has 2j-5 or fewer observations because $d(P_{2n-5}, P_{2n})$ is not an observation in N_{2n} although it is an observation in $N_{2(n-1)}$. Hence, A has at most 2k - 4 observations.

Case 3: j = 2(n-1). In this case B equals N_{2n}* which has 2j - 4 observations, one less than $N_{2j} = N_{2(n-1)}$. A equals B, or A is formed from B by adding P_{2n-2} or P_{2n-1} but not both. If A equals B, then j = k and A has 2k - 4 observations. If A is formed from B by adding P_{2n-2} or P_{2n-1}, then k = j + 1 and A has at most 2 more observations than B. Hence, A has 2k - 4 observations. This completes the proof of theorem 4.

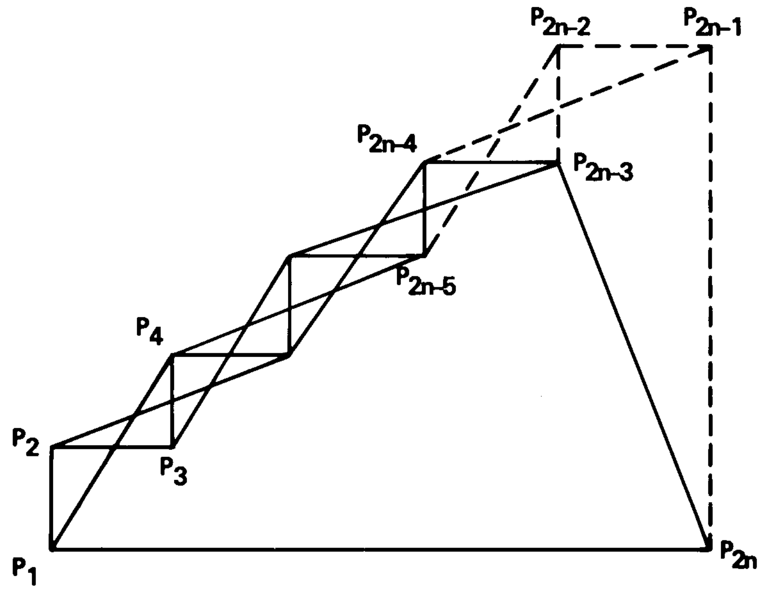

Figure 10.--The general staircase network. N_{2n} corresponds to all lines. N_{2n}^* corresponds to all solid lines. $N_{2(n-1)}$ is equivalent to N_{2n}^* with an additional line between P_{2n-5} and P_{2n}.

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