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An Advantageous, Alternative Parameterization of Rotations for Analytical Photogrammetry

ALLEN J. POPE

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An Advantageous, Alternative Parameterization of Rotations for Analytical Photogrammetry

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ABSTRACT. A case is made for increased use of a method of representing an orthogonal matrix that is different from the one now used in most analytical photogrammetric solutions. The relevant computational formulas are given, along with their derivation and geometric interpretation.

INTRODUCTION

Recent proposals (Brown 1968, Koch 1970) for photogrammetric mapping of the moon and planets involve the analytical combination of thousands of photographs which it will be desirable to adjust in as large simultaneous groups as possible. Earth-bound photogrammetrists continue to strive for the capability of practical simultaneous adjustment of ever larger blocks (Keller 1967). In these undertakings even small savings of computer time per photogram are worth considering because of the large number of photographs and the desirability of iterating to a least-squares solution of the true nonlinear problem. It is, admittedly, unlikely that changes in the parameterization of the basic equations will be attractive to those already in possession of a workable program adequate for problems in hand. The opportunity for incorporation of such alternatives at an appropriately early developmental state presents itself in the approaching generation of photogrammetric problems.

Any use of the phrase "new parameters" below is for convenience of designation only. The underlying results are in fact very old, attributable to Euler, Cayley, Hamilton, Rodrigues, Gauss, and others. Similar alternative parameterizations of rotations have been surveyed in the photogrammetric literature by Schut (1959), who has given the basic equations and pointed out some applications (Schut 1961). Similar parameterizations have been considered previously by Rinner (1957) and Allen (1954–55). This report aims to extend the consideration of such an alternative parameterization to include the exact manner of its integration into a large simultaneous analytical adjustment procedure, using observation equations and Newton-Gauss (Taylor series) iteration.

It has to be conceded that this alternative parameterization is not as intuitively simple to formulate and visualize as is the usual parameterization, for example, in terms of ϕ, ω, κ, in spite of the simplicity and attractiveness of the final computational formulas. This is partly due to the lack of a unified exposition of the relevant proofs and geometric interpretations underlying these parameters and aimed at the photogrammetric user. In fact, potential photogrammetric users may be discouraged because of a certain amount of algebraic "messiness" and complexity apparently inherent in this parameterization in contrast to the conceptual simplicity of the conventional ϕ, ω, κ approach. It is believed desirable to understand why the computational formulas work before they are used and that such understanding will inevitably lead to their increased use. Thus, another goal of this report is to give such a unified exposition using modern notation and avoiding algebraic "messiness," where possible.

A third aim of this report is to draw a connection between the proposed alternative parameterization and an increasingly used formulation of differential rotations (Lucas 1963, Brown and Trotter 1969, Gyer, Lewis, and Saliba 1967). The parameterization considered here may be viewed as a logical extension of this line of thought. Because of this, a discussion of differential rotations in the conventional parameterization, while not a necessary part of a minimal description of the alternative parameterization, is given to provide explanatory parallels and useful auxiliary formulas. The basic ideas relating to differential rotations are found in Frazer, Duncan, and Collar (1938).

PRELIMINARY DEFINITIONS

Various notation and sign conventions need to be established at this point. Starting with the colinearity equations in vector form, we have

$$\begin{bmatrix} x \\ y \\ c \end{bmatrix} = sM \begin{bmatrix} X-X_0 \\ Y-Y_0 \\ Z-Z_0 \end{bmatrix}, \tag{1}$$

where x and y are plate coordinates of an image with respect to an origin at the principal point, and c is the focal length. X, Y, Z are the ground coordinates of the object whose image is at (x, y); X_0, Y_0, Z_0 are the coordinates of the camera station; and s is an initially unknown scale. Dividing the first and second equations by the third to eliminate s, we have the colinearity equations in the conventional, divided, solved form

$$\begin{cases} x = c\dfrac{p}{r} \\ \quad, \\ y = c\dfrac{q}{r} \end{cases} \tag{2a}$$

where

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = M \begin{bmatrix} X-X_0 \\ Y-Y_0 \\ Z-Z_0 \end{bmatrix}. \tag{2b}$$

M, the main object of interest in this report, is a 3 by 3 orthogonal matrix for which the conventional photogrammetric parameterization is a special case of

$$M = R_i(\alpha) R_j(\beta) R_k(\gamma) \tag{3}$$

for some choice of (i, j, k) and identification of (α, β, γ). The particular choice varies and is essentially immaterial, although specific examples are considered for illustration.

In fact, another underlying motivation of the approach to rotations discussed here is to show that many interesting and useful formulas, in fact the *entire analytical adjustment apparatus*, can be obtained *without any commitment* to a particular choice of (i, j, k). The choice, along with the choice of primary, secondary, and tertiary orientation angles (α, β, γ), is viewed as a nuisance inherited from analog photogrammetry. In the case that a particular set of angles, such as ϕ, ω, κ, is of physical interest, the angles can still be easily obtained from the alternative parameters by the method given in Transformation of Parameters, below. One of the motivations for writing this report has been to present general formulas that establish the exact and differential relations between various parameters.

Each successive matrix R_l represents a rotation of the coordinate system about the current l axis. For later reference, the R_l's are spelled out:

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \tag{4a}$$

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{4b}$$

$$\text{and } R_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4c}$$

For example, in equation (3) for $(i, j, k) = (3, 2, 1)$ and $(\alpha, \beta, \gamma) = (\kappa, \omega, \phi)$ we have

$$M = \begin{bmatrix} \cos\kappa\cos\omega & \sin\kappa\cos\phi + \cos\kappa\sin\omega\sin\phi & \sin\kappa\sin\phi - \cos\kappa\sin\omega\cos\phi \\ -\sin\kappa\cos\omega & \cos\kappa\cos\phi - \sin\kappa\sin\omega\sin\phi & \cos\kappa\sin\phi + \sin\kappa\sin\omega\cos\phi \\ \sin\omega & -\cos\omega\sin\phi & \cos\omega\cos\phi \end{bmatrix}. \tag{5}$$

The axes are understood to be numbered 1, 2, 3, corresponding to x, y, z; the senses of the rotations R_3, R_1, R_2 are 1 to 2, 2 to 3, 3 to 1, respectively, for positive θ, whether the system be right- or left-handed. This specification is independent of the viewpoint of the observer. To recapitulate, a rotation in the sense of R_l would cause a right-handed screw to advance along the positive l axis in a right-handed system and a left-handed screw to advance along the positive l axis in a left-handed system. Viewed from the positive "end" of the l axis, the rotation of the coordinate axes is counterclockwise in a right-handed system and clockwise in a left-handed system. Retention of the same convention (1 to 2, 2 to 3, 3 to 1, for $+\theta$) to describe the rotation of a rigid body in a fixed coordinate system, rather than a rotation of the coordinate system producing new coordinates of fixed points, leads to replacement of θ by $-\theta$ everywhere in formulas (4a), (4b), and (4c).

The parameterization of equation (3) represents all possible *proper orthogonal* 3 by 3 matrices; that is, having det $M=+1$ and (equivalently) producing no change of handedness. It frequently happens in practice that the handedness of the ground coordinate system and the camera coordinate system are different, in which case M must be extended by the inclusion of one or more permutations, reflections, or inversions—represented, for example, by matrices like

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

respectively, giving det $M=-1$. For the most part, it is possible to assume that det $M=+1$ without loss of generality by absorbing the change of handedness into the equations before or after application of M.

The case of $(i, j, k)=(3, 1, 3)$ or $(3, 2, 3)$, the "Eulerian angles" of physics and astronomy, is seldom used in photogrammetry but is mentioned at one or two points below for comparison purposes.

Distortion corrections and models are not discussed in this technical report. The photogrammetric model is taken to be completely given by equation (2). Alternatives to the observation equation approach to the least-squares solution are not considered.

The u unknown parameters (X, Y, Z at all unknown ground points, X_0, Y_0, Z_0, α, β, γ for all cameras) are denoted by vector $\bar{\beta}$, such that $\sum_{i=1}^{n} (v_{x_i}^2+v_{y_i}^2)$ is a minimum (where $v_{x_i}=x_i(\bar{\beta})-x_i^{\text{obs}}$, similarly for v_{y_i}, and $n(>u)$ is the number of measured points).

On the basis of equations (2a) and (2b) the Newton-Gauss iterative solution of the least-squares problem of finding $\bar{\beta}$ proceeds in outline as follows: The nonlinear functions $x(\bar{\beta})$ and $y(\bar{\beta})$, given in (2a) and (2b), are replaced by the linear terms of a Taylor series expansion about some preliminary approximate values $\bar{\beta}_0$ of the unknown parameters $\bar{\beta}$. That is, $x(\bar{\beta})=x(\bar{\beta}_0)+\left.\frac{\partial x}{\partial \bar{\beta}}\right|_{\bar{\beta}_0}\Delta\bar{\beta}$, where $\frac{\partial x}{\partial \bar{\beta}}$ is a row vector of partial derivatives evaluated at $\bar{\beta}=\bar{\beta}_0$ and $\Delta\bar{\beta}=\bar{\beta}-\bar{\beta}_0$. Thus we have v_i^0 (linearized) $=\{x_i(\bar{\beta}_0)-x_i^{\text{obs}}\}+\left.\frac{\partial x}{\partial \bar{\beta}}\right|_{\bar{\beta}_0}\Delta\bar{\beta}$. Now $\Delta\bar{\beta}_{LS}$ which minimizes $\bar{V}^{0\,t}P\bar{V}^0$ (linearized) can be solved for by the usual apparatus of linear least squares; formation of normal equations; etc. Then $\bar{\beta}_1=\bar{\beta}_0+\Delta\bar{\beta}_{LS}^0$, and the process is repeated with $\bar{\beta}_1$ as the n-dimensional point of expansion for the Taylor series. Next $\Delta\bar{\beta}_{LS}^1$, minimizing the sums of squares of linearized residuals $v_{x_i}^1$, is found. Repetition of this procedure leads, hopefully, since convergence cannot be guaranteed in general, to converge to values β which minimize the sum of squares of nonlinear residuals.

The main feature of interest here is that each recomputation of $x(\bar{\beta}_l)$ and $\left.\frac{\partial x}{\partial \bar{\beta}}\right|_{\bar{\beta}_l}$ in the conventional formulation necessitates the recomputation of M (e.g., by equation (5)) and the partials of M with respect to α, β, and γ (detailed below). Any new parameters must yield competitive formulas for (a) computation of partials and (b) "updating" of M. From the point of view of a strictly analytical approach to photogrammetry, formulas for (a) and (b) constitute the whole story of any parameterization of M. Additional proofs and geometric interpretations are desirable for better understanding, but not strictly necessary for a minimal workable solution. Because of this and to give a concrete basis for subsequent discussion, the yet-to-be-derived formulas for (a) and (b) are given below as formulas (9) through (11). Their form will immediately support the claimed advantages of the alternative parameterization.

COMPUTATIONAL FORMULAS

In the discussion of (a), that is to say, the coefficients in the differential forms for dx, dy in terms of dX, dY, dZ, dX_0, dY_0, dZ_0 and the differential changes in the parameters of M, it is convenient to group the partials in the following manner:

Denote

$$A=\begin{bmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} & \frac{\partial y}{\partial r} \end{bmatrix}=\frac{c}{r}\begin{bmatrix} 1 & 0 & -\frac{p}{r} \\ 0 & 1 & -\frac{q}{r} \end{bmatrix},$$

so that

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = A \begin{bmatrix} dp \\ dq \\ dr \end{bmatrix}$$

$$= A \left\{ (dM) \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix} + M \begin{bmatrix} dX - dX_0 \\ dY - dY_0 \\ dZ - dZ_0 \end{bmatrix} \right\}. \quad (6a)$$

This grouping is useful in discussing both old and new parameterizations. Note that differentiation of $MM^t = I$ yields $dMM^t + MdM^t = 0$, so that $(dMM^t) = -(dMM^t)^t$. If $A^t = -A$, A is said to be a skew-symmetric matrix. The most general 3 by 3 skew-symmetric matrix can be written in the form

$$S_{\bar{\omega}} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix},$$

where $S_{\bar{\omega}}$ denotes a skew-symmetric matrix using the components of the vector $\bar{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$. Thus

$$dMM^t = S_{\bar{\omega}} \quad (7a)$$

and

$$dM = S_{\bar{\omega}} M \quad (7b)$$

in advance of any particular choice of parameters for M. An exactly similar consideration starting with $M^tM = I$ gives the alternate but unused form $dM = MS_{\bar{\omega}}$. In (7b) $\bar{\omega}$ is some as-yet-unknown linear function of the differential changes in the parameters chosen to represent M. These functions can be obtained directly from equation (7a) or, more easily, by the convenient general formulas given below for the conventional parameterizations. In the new parameterization, the components of $\bar{\omega} - \omega_1, \omega_2, \omega_3 -$ are themselves the unknowns in the observation equations, and no further substitution is needed.

Inserting equation (7b) into (6a) and using equation (2b) along with the identities $S_{\bar{\omega}}\bar{v} = -\bar{\omega} \times \bar{v} = \bar{v} \times \bar{\omega} = -S_{\bar{v}}\bar{\omega}$, where $\times$ indicates the vector cross product, that is,

$$\bar{\omega} \times \bar{v} = \begin{bmatrix} \omega_2 r - \omega_3 q \\ \omega_3 p - \omega_1 r \\ \omega_1 q - \omega_2 p \end{bmatrix} \text{ where } \bar{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \text{ and } \bar{v} = \begin{bmatrix} p \\ q \\ r \end{bmatrix},$$

we have

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = L \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + AM \begin{bmatrix} dX - dX_0 \\ dY - dY_0 \\ dZ - dZ_0 \end{bmatrix}, \quad (6b)$$

where

$$L = -As_{\bar{v}} \text{ and } S_{\bar{v}} = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix}.$$

That is,

$$L = \frac{c}{r} \begin{bmatrix} pq/r & -(r^2+p^2)/r & q \\ (r^2+q^2)/r & -pq/r & -p \end{bmatrix}, \quad (8a)$$

the truly vertical coefficients. For convenience, p, q, r are often approximated at the first stage of the iteration using the observed x and y. That is, (8a) is replaced with

$$L = \begin{bmatrix} xy/c & -(c^2+x^2)/c & y \\ (c^2+y^2)/c & -xy/c & -x \end{bmatrix}. \quad (8b)$$

Whether or not this approximation is used at the first evaluation of L, care must be taken to evaluate L in subsequent cycles of the iteration using (8a), or, what is the same thing, (8b) using the most recent adjusted plate coordinates x and y. Failure to do this will, in general, preclude convergence of the iteration to the nonlinear least-squares solution.

Then, the proposed new parameterization is contained in the following equations:

Denote by q the 4-tuple $(\delta, \alpha, \beta, \gamma)$. The iteration may be started with $M_0 = I$, the 3 by 3 unit matrix, and with $q_0 = (1, 0, 0, 0)$. If any auxiliary data are available that can be used to establish a better initial orientation, the methods discussed below under Transformation of Parameters can be used to compute q_0.

Observation equations based on (6b) and (8a) lead to a least-squares solution for ω_1, ω_2, ω_3 and differential corrections to other parameters dX, dY, . . . at each stage of the Newton-Gauss iteration. From the current values of ω_1, ω_2, ω_3, and the current values of the parameters $q_i = (\delta_i, \alpha_i, \beta_i, \gamma_i)$, a new set of parameters, q_{i+1}, is computed, using

$$\begin{cases} \delta_{i+1} = \delta_i - \alpha_i\bar{\omega}_1 - \beta_i\bar{\omega}_2 - \gamma_i\bar{\omega}_3 & \bar{\omega}_1 = \omega_1/2 \\ \alpha_{i+1} = \alpha_i + \delta_i\bar{\omega}_1 - \gamma_i\bar{\omega}_2 + \beta_i\bar{\omega}_3 & \bar{\omega}_2 = \omega_2/2 \\ \beta_{i+1} = \beta_i + \gamma_i\bar{\omega}_1 + \delta_i\bar{\omega}_2 - \alpha_i\bar{\omega}_3 & \bar{\omega}_3 = \omega_3/2 \\ \gamma_{i+1} = \gamma_i - \beta_i\bar{\omega}_1 + \alpha_i\bar{\omega}_2 + \delta_i\bar{\omega}_3 & \end{cases} \quad (9)$$

(15 products) and with

$$\begin{aligned} l &= \delta^2+\alpha^2+\beta^2+\gamma^2 \\ l_1 &= 1/l \\ l_2 &= 2l_1 \\ \bar{\alpha} &= l_2\alpha \\ \bar{\beta} &= l_2\beta \\ \bar{\gamma} &= l_2\gamma \end{aligned} \tag{10}$$

(8 products and 1 division), the updated M is computed from

$$M=\begin{bmatrix} (\delta^2+\alpha^2-\beta^2-\gamma^2)l_1 & \alpha\bar{\beta}+\delta\bar{\gamma} & \alpha\bar{\gamma}-\delta\bar{\beta} \\ \alpha\bar{\beta}-\delta\bar{\gamma} & (\delta^2-\alpha^2+\beta^2-\gamma^2)l_1 & \beta\bar{\gamma}+\delta\bar{\alpha} \\ \alpha\bar{\gamma}+\delta\bar{\beta} & \beta\bar{\gamma}-\delta\bar{\alpha} & (\delta^2-\alpha^2-\beta^2+\gamma^2)\,l_1 \end{bmatrix} \tag{11}$$

(9 additional products, utilizing repeated terms).

Thus, M has been parameterized in terms of the 4-tuple $q=(\delta, \alpha, \beta, \gamma)$. We shall see that, as expected, only three are independent, for example, α, β, γ. The "extra" dependent parameter δ is retained to produce more convenient formulas. Note that α, β, γ in the new parameterization are *not* the same as α, β, γ in the conventional parameterization of equation (3). Formulas (9), (10), and (11) can be made more compact by obvious substitutions, but the forms given show a sequence of operations that avoids needless repetitions of products.

The advantages of this parameterization can now be appreciated by considering equations (9) through (11) in comparison with equation (3).

1. No trigonometric functions and no square roots are needed.
2. The total count of operations is small—32 products and 1 division.
3. Particular choices of $(i, j, k,)$ are unnecessary. Since in a complex problem it is not unusual to find several choices coexisting (e.g., different for exterior and interior orientations), this circumvents a potential source of confusion.
4. The parameterization is "more nearly linear" than the conventional one. M is only quadratic in (normalized) q, whereas the formula for any trig function involves an infinite sum of powers. In fact, Schut (1961) has shown that a closely related representation of a rotation (equation (28) below), leads to a linear solution in some problems when compromises with the rigorous assignment of residuals are made (use of equation (28b) does not lead to a linear solution in the rigorously formulated general analytical least-squares solution, however). If an alternative, more nearly linear parameterization exists, its use will lead to improved convergence of the Newton-Gauss, Taylor series iteration. A simple example to illustrate the point is the least-squares solution of

$$v_{x_i}+x_i'=s\cos\theta x_i+s\sin\theta y_i$$

$$v_{y_i}+y_i'=-s\sin\theta x_i+s\cos\theta y_i$$

$$i=1,\ldots,n$$

for fixed x_i and y_i. This parameterization is nonlinear and when solved by the Newton-Gauss method (not strictly necessary for this simple problem), must be iterated to produce the least-squares solution for s and θ. On the other hand, use of the equivalent parameterization

$$\begin{aligned} x_i' &= ax_i+by_i \\ y_i' &= -bx_i+ay_i \end{aligned}$$

gives a linear, one-step solution for a and b, and requires no approximate initial values. This is an extreme example of increased rate of convergence!

The fact that the parameterization is only quadratic suggests a direct rather than the Taylor series approach to an iterative solution—that is, moving small second-order terms to the opposite side, etc., which may be practical for resections, at least. This line of thought is not pursued in this report.

5. The coefficients of the observation equation to be normalized and solved for the $(\omega_1, \omega_2, \omega_3)$'s and $(dX, dY, dZ, dX_0, dY_0, dZ_0)$'s at each iteration are simplified. In fact, they always have essentially the functional form which, in the conventional approach, is associated only with the truly vertical case. This is seen from equations (6b) and (8a). The observation equations for the conventional parameterization differ from these (in the nontruly vertical case) by the presence of an additional substitution of the form $\overline{\omega}=C\begin{bmatrix} d\kappa \\ d\omega \\ d\phi \end{bmatrix}$, where the coefficients in the 3 by 3 matrix C are functions of the current

values of ω and κ that are detailed in the next section.

6. Finally, the above-suggested procedure shares the advantages of a method of handling differential rotations in photogrammetry which will be next discussed.

DIFFERENTIAL FORMS FOR THE CONVENTIONAL PARAMETERIZATION

From equation (3) we have (see Frazer, Duncan, and Collar 1938)

$$dM = \mathring{R}_i R_j R_k d\alpha + R_i \mathring{R}_j R_k d\beta + R_i R_j \mathring{R}_k d\gamma,$$

where $\mathring{R}_l$ represents $\dfrac{\partial R_l(\theta)}{\partial \theta}$ $l = i, j, k$. Since $MM^t = I$, $R_l R_l^t = I$, and $M^t = R_k^t R_j^t R_i^t$, dM can be written

$$dM = \{S_i d\alpha + R_i S_j R_i^t d\beta + R_i R_j S_k R_j^t R_i^t d\gamma\} M,$$

where $S_l = \mathring{R}_l R_l^t$, $l = i, j, k$, are skew-symmetric matrices having the form $S_l = S_{\bar{\epsilon}_l}$. Here $\bar{\epsilon}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\bar{\epsilon}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\bar{\epsilon}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$. The notation $S_{\bar{r}}$, $\bar{r}$ a vector, means $S_{\bar{r}} = \begin{bmatrix}0 & r_3 & -r_2\\ -r_3 & 0 & r_1\\ r_2 & -r_1 & 0\end{bmatrix}$, as before. This is confirmed by direct computation of the products $\mathring{R}_l R_l^t$.

The following properties hold for skew-symmetric matrices:

(a) $kS_{\bar{r}} = S_{k\bar{r}}$, k a scalar

(b) $S_{\bar{r}_1} + S_{\bar{r}_2} = S_{(\bar{r}_1 + \bar{r}_2)}$

(c) $MS_{\bar{r}}M^t = S_{\bar{r}'}$ where $\bar{r}' = M\bar{r}$, M proper orthogonal

(d) $A^t S_{\bar{r}} A = S_{\bar{r}'}$ where $\bar{r}' = \tilde{A}\bar{r}$ for arbitrary square A. Here $\tilde{A}$ = (adjoint matrix of A) = (det A) A^{-1} for non-singular A.

Formulas (a) and (b) can be directly confirmed. The transformation formula (d), of which (c) is a special case, can be derived by direct expansion and use of the triple inner product $A \cdot (B \times C)$ as follows: Denoting the lth column of A by $\bar{a}_l$, the ijth element of $A^t S_{\bar{r}} A$ is $-\bar{a}_i \cdot (\bar{r} \times \bar{a}_j) = (\bar{a}_i \times \bar{a}_j) \cdot \bar{r}$. Note that $\bar{a}_i \times \bar{a}_i = 0$ and $\bar{a}_i \times \bar{a}_j = -\bar{a}_j \times \bar{a}_i$, so that $A^t S_{\bar{r}} A$ is directly seen to be skew-symmetric, a fact already known from application of the definition $S^t = -S$ to $A^t S_{\bar{r}} A$. The elements of $S_{\bar{r}'} = A^t S_{\bar{r}} A$ are

$$r_1' = (\bar{a}_2 \times \bar{a}_3) \cdot \bar{r}$$
$$r_2' = -(\bar{a}_1 \times \bar{a}_3) \cdot \bar{r}$$
$$r_3' = (\bar{a}_1 \times \bar{a}_2) \cdot \bar{r}$$

or

$$\bar{r}' = \begin{bmatrix}(\bar{a}_2 \times \bar{a}_3)^t\\ -(\bar{a}_1 \times \bar{a}_3)^t\\ (\bar{a}_1 \times \bar{a}_2)^t\end{bmatrix} \bar{r};$$

but this matrix is the adjoint of A. A lengthy proof of (c) is found in Goldstein (1950, page 118) and (d) allows extension of the following results to improper rotations containing permutations, reflections, and/or inversions.

Using these results in the order (c), (a), (b), we see that

$$dM = S_{\bar{r}} M, \tag{12}$$

$S_{\bar{r}}$ as before, and $\bar{r} = \bar{\epsilon}_i d\alpha + R_i \bar{\epsilon}_j d\beta + R_i R_j \bar{\epsilon}_k d\gamma$ (note similarity to equation (7) above). This can be written as

$$\bar{r} = C \begin{bmatrix}d\alpha\\ d\beta\\ d\gamma\end{bmatrix}, \tag{13}$$

where

$$C = [\bar{\epsilon}_i \,|\, R_i \bar{\epsilon}_j \,|\, R_i R_j \bar{\epsilon}_k]. \tag{14}$$

(Note $R_l \bar{\epsilon}_l = \bar{\epsilon}_l$ so that $R_i R_j \bar{\epsilon}_k = M \bar{\epsilon}_k$. Also, note that $A\bar{\epsilon}_k$ is the kth column of A for any 3 by 3 A.)

An exactly similar procedure will yield the alternate form $dM = MS_{\bar{r}'}$ *where*

$$\bar{r}' = M^t \bar{r} = C' \begin{bmatrix}d\alpha\\ d\beta\\ d\gamma\end{bmatrix}$$

and

$$C' = [R_k^t R_j^t \bar{\epsilon}_i \,|\, R_k^t \bar{\epsilon}_j \,|\, \bar{\epsilon}_k].$$

This is also seen directly from $dM = M(M^t S_{\bar{r}} M)$ and (c) above.

C is a 3 by 3 matrix. For example, in the case of (3, 2, 1), (κ, ω, ϕ) above

$$C = \begin{bmatrix}0 & \sin\kappa & \cos\kappa\cos\omega\\ 0 & \cos\kappa & -\sin\kappa\cos\omega\\ 1 & 0 & \sin\omega\end{bmatrix}. \tag{15}$$

The important point about equation (13) in this discussion is that it constitutes an exact linear substitution with a unique inverse solution, except in the case det $C=0$. That is,

$$\begin{bmatrix} d\alpha \\ d\beta \\ d\gamma \end{bmatrix} = C^{-1}\bar{r}. \tag{16}$$

An easily proved invariance property of linear least squares is that the values of $d\alpha$, $d\beta$, $d\gamma$, obtained from a least-squares solution of observation equations in which $d\alpha$, $d\beta$, $d\gamma$ appear explicitly as unknowns, are equal to the values of $d\alpha$, $d\beta$, $d\gamma$ computed by equation (16) from values of r_1, r_2, r_3. These are obtained from a least-squares solution based on observation equations in which (13) was not substituted, so that r_1, r_2, r_3 appear explicitly as unknowns. The latter procedure is attractive because it yields observation equations that are simpler by the omission of all C matrices (note that

$$M\begin{bmatrix} dX_0 \\ dY_0 \\ dZ_0 \end{bmatrix}$$

is also such an exact linear substitution, although in the general problem

$$M\begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix}$$

is not, since ground points are seen by more than one camera, and therefore dX, dY, dZ have more than one M as coefficient). There is an analogous invariance property for nonlinear least-squares problems. Although appropriate, it is not used here, since each stage of the Newton-Gauss method is a linear least-squares problem.

Although the computation of C^{-1}, numerically or algebraically, presents no real problem and we find that the q parameterization avoids the need, it is of interest in this connection to note the following explicit forms for C^{-1}. Two cases are distinguished:

1. (i, j, k) distinct:

$$C^{-1} = BE^tR_i^t, \tag{17}$$

where $B=\begin{bmatrix} 1 & 0 & s\tan\beta \\ 0 & 1 & 0 \\ 0 & 0 & \sec\beta \end{bmatrix}$, $E=[\bar{\epsilon}_i|\bar{\epsilon}_j|\bar{\epsilon}_k]$, β is the angle associated with R_j as in equation (3), and

$$s=\begin{cases} +1 & \text{for } (i,j,k) = \text{a cyclic permutation of } (1,2,3) \\ -1 & (i,j,k) \neq \text{a cyclic permutation of } (1,2,3). \end{cases}$$

Also, note that det $C=\cos\beta$.

2. (k, j, k): In practice, only (3, 2, 3) or (3, 1, 3) are considered (Eulerian angles), and the two are related by

$$R_3(\alpha)R_2(\beta)R_3(\gamma) = R_3(\alpha-\pi/2)R_1(\beta)R_3(\gamma+\pi/2)$$
$$= R_3(\alpha+\pi/2)R_1(-\beta)R_3(\gamma-\pi/2).$$

Now define s by $s\bar{\epsilon}_i=\bar{\epsilon}_j\times\bar{\epsilon}_k$.

Then

$$C^{-1}=\bar{B}E^tR_k^t, \qquad \text{where } \bar{B}=\begin{bmatrix} s\cos\beta & 0 & 1 \\ 0 & 1 & 0 \\ -s\sec\beta & 0 & 0 \end{bmatrix},$$

and det $C=s\sin\beta$.

As an example, formula (17) applied to the same case of equation (15) yields

$$C^{-1}=\begin{bmatrix} -\cos\kappa\tan\omega & \sin\kappa\tan\omega & 1 \\ \sin\kappa & \cos\kappa & 0 \\ \cos\kappa\sec\omega & -\sin\kappa\sec\omega & 0 \end{bmatrix}.$$

It can happen, in unusual but possible cases, that det $C=0$, so that C^{-1} does not exist. The differential correction formulas for the q parameters avoid this problem.

Results similar to the above have been used by photogrammetrists in (a) simulation studies, (b) resection solutions, and (c) in derivations of differential relations between alternate choices of (i, j, k) and (α, β, γ). For the general analytic problem using Newton-Gauss iteration and a conventional parametrization like equation (3), it is still necessary to compute at *each* iteration as many C's (or C^{-1}'s) as there are cameras in order to get $\begin{bmatrix} d\alpha \\ d\beta \\ d\gamma \end{bmatrix}$ for addition to α, β, γ. α, β, γ are then used in an equation like (5) to compute the new M needed in the coefficients and constant terms of the observation equations. It makes little difference in the number of operations whether the C's or C^{-1}'s are computed before or after the linear least-squares solution at each stage.

This undoubtedly explains why most large analytical solutions are still done in terms of $d\phi$, $d\omega$, $d\kappa$ or similar angles, and why no attempt is made to take advantage of the simpler form of the observation equations obtained by omitting C. A genuine saving results if one can bypass the computation of C and/or C^{-1} altogether and compute an orthogonal, updated M directly as a function of r_1, r_2, r_3 (recognizable as equivalent to ω_1, ω_2, ω_3 of equation (7b)). The q parameters achieve this to a surprising degree.

DISCUSSION OF THE ALTERNATIVE PARAMETERIZATION

In line with the motivations mentioned above, this discussion is divided into five sections: derivation and interpretation of the basic formulas and various auxiliary formulas; development of differential forms for the new parameterization; leading into the choice of formulas for use in the photogrammetric application; the general method for obtaining, and examples of, formulas which make possible inverse and direct transformation from the new parameters to other parameterizations; and a small numerical example.

Derivation and Interpretation of Formulas

The following development has as its goal a concise derivation of all the formulas needed in a photogrammetric application of the q parameterization. In addition, several nonessential formulas and various geometric interpretations will be obtained. All of these contribute to a better understanding of the q parameterization and an appreciation of its close connection to various alternatives, which at first sight may appear unrelated.

Surprisingly, it is much easier to obtain the desired formulas for 3 by 3 orthogonal matrices by using a particular kind of 4 by 4 orthogonal matrix, which is now defined. Use of 4 by 4 matrices is not a necessary part of the proof, but it greatly simplifies and clarifies the algebra involved. With $q=(\delta, \alpha, \beta, \gamma)$, real values, as before, define

$$P_q=\begin{bmatrix} \delta & \alpha & \beta & \gamma \\ -\alpha & \delta & \gamma & -\beta \\ -\beta & -\gamma & \delta & \alpha \\ -\gamma & \beta & -\alpha & \delta \end{bmatrix},$$

$$Q_q=\begin{bmatrix} \delta & -\alpha & -\beta & -\gamma \\ \alpha & \delta & \gamma & -\beta \\ \beta & -\gamma & \delta & \alpha \\ \gamma & \beta & -\alpha & \delta \end{bmatrix}, \qquad (18)$$

and $\nu=(\delta^2+\alpha^2+\beta^2+\gamma^2)^{1/2}>0$. It is easily verified that $P_q^tP_q=Q_q^tQ_q=\nu^2I$, so that $P_{\hat{q}}=\frac{1}{\nu}P_q$ and $Q_{\hat{q}}=\frac{1}{\nu}Q_q$ are orthogonal 4 by 4 matrices.

If $\nu^2=1$, q is said to be normalized and is denoted $\hat{q}$. A given q can always be normalized by dividing by ν; $\hat{q}=q/\nu$. A normalized $\hat{q}$ contains only three independent parameters—arbitrarily taken to be α, β, γ. The fourth component of $\hat{q}$, δ, is determined from α, β, γ by the constraint $\delta^2+\alpha^2+\beta^2+\gamma^2=1$. The notation will usually indicate whether one is dealing with a normalized $\hat{q}$ or unnormalized q, as one or the other becomes more convenient.

Since $P_{\hat{q}_1}$ and $Q_{\hat{q}_2}$ are orthogonal, so is $T_{\hat{q}_1,\hat{q}_2}=P_{\hat{q}_1}Q_{\hat{q}_2}$. In fact $T_{\hat{q}_1,\hat{q}_2}$ is the most general 4 by 4 proper orthogonal matrix, for in $\hat{q}_1$ and $\hat{q}_2$ there are 6 independent parameters (the 10 independent orthogonality conditions contained in $T^tT=I$ imposed on the 16 elements of T leave 6 independent parameters to be specified). This fact is of only passing interest here, however. Of more interest is the fact that the most general proper orthogonal 3 by 3 is obtained from a special case of $T_{\hat{q}_1,\hat{q}_2}$.

With $\hat{q}_1=\hat{q}_2=\hat{q}$, denote $T_{\hat{q}_1,\hat{q}_2}=T_{\hat{q}}$. Performing the multiplication $T_{\hat{q}}=P_{\hat{q}}Q_{\hat{q}}$, we find

$$T_{\hat{q}}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & M & \\ 0 & & & \end{bmatrix},$$

where M is a 3 by 3 matrix whose terms are found to be

$$M=\begin{bmatrix} \delta^2+\alpha^2-\beta^2-\gamma^2 & 2(\alpha\beta+\gamma\delta) & 2(\alpha\gamma-\beta\delta) \\ 2(\alpha\beta-\gamma\delta) & \delta^2-\alpha^2+\beta^2-\gamma^2 & 2(\beta\gamma+\alpha\delta) \\ 2(\alpha\gamma+\beta\delta) & 2(\beta\gamma-\alpha\delta) & \delta^2-\alpha^2-\beta^2+\gamma^2 \end{bmatrix}. \qquad (19a)$$

Since $T_{\hat{q}}$ is orthogonal,

$$I = T_{\hat{q}}^{t} T_{\hat{q}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & M^{t}M & \\ 0 & & & \end{bmatrix}$$

or

$$M^{t}M = I,$$

so that M is orthogonal. Direct expansion of det P_q and det Q_q yields det $P_q = \det Q_q = \nu^4 > 0$. Also, since det $M = \det T_{\hat{q}} = (\det P_{\hat{q}}) \times (\det Q_{\hat{q}})$, and det $P_{\hat{q}} = \det Q_{\hat{q}} = +1$ because $\nu^2 = 1$, we have

$$\det M = +1$$

so that M is proper orthogonal, that is, includes no change of handedness. Only proper orthogonal 3 by 3's are parameterized by equation (19a) for real parameters, and changes of handedness are introduced when necessary by combining M with one or more inversions, reflections or permutations, as described above (page 3). If q is unnormalized, it is easily seen that an orthogonal M is given by

$$M = \frac{1}{\nu^2} \begin{bmatrix} \delta^2 + \alpha^2 - \beta^2 - \gamma^2 & 2(\alpha\beta + \gamma\delta) & 2(\alpha\gamma - \beta\delta) \\ 2(\alpha\beta - \gamma\delta) & \delta^2 - \alpha^2 + \beta^2 - \gamma^2 & 2(\beta\gamma + \alpha\delta) \\ 2(\alpha\gamma + \beta\delta) & 2(\beta\gamma - \alpha\delta) & \delta^2 - \alpha^2 - \beta^2 + \gamma^2 \end{bmatrix}, \tag{19b}$$

which is the same as (19a) with the insertion of a factor $1/\nu^2$. The computational significance of (19b) is that the presence of ν^2 enables the square root to be avoided that would otherwise be involved in a normalization using division by ν.

That M defined in (19a) is the most general 3 by 3 proper orthogonal matrix follows from the fact that the 6 independent orthogonality conditions in $M^{t}M = I$ imposed upon the 9 elements of M leave 3 independent parameters to be specified, and M is defined in terms of $\hat{q}$ containing 3 independent parameters. On the other hand, (19b) contains four parameters so that there are an infinite number of sets of δ, α, β, γ (unnormalized) that produce a single orthogonal M by (19b). In (19a), notice that $\hat{q}$ and $-\hat{q}$ produce the same M, a fact whose geometric meaning will later become apparent. Other than this, the correspondence between proper orthogonal matrices M and the three independent elements of normalized $\hat{q}$ defined by equation (19a) is one-to-one.

The use of the 4 by 4 orthogonal forms P and Q has allowed us to "discover" and prove the q parameterization of equation (19a). In a similar manner, use of P and Q lead to "discovery" and proof of other useful formulas.

From equation (18) for P and Q, note that $P_q^t = P_{\bar{q}}$ and $Q_q^t = Q_{\bar{q}}$, where $\bar{q} = (\delta, -\alpha, -\beta, -\gamma)$, $q = (\delta, \alpha, \beta, \gamma)$. $\bar{q}$ is called the hypercomplex conjugate of q because in the i, j, k representation discussed below it is the analogy of the conjugate of a complex number.

The product $P_{q_2} P_{q_1}$ is found to be another matrix of the form P_{q_3}, and the product $Q_{q_2} Q_{q_1}$ can also be verified to result in another matrix of the form Q_{q_3} where, in both cases, q_3 is given by the equations

$$\begin{aligned} \delta_3 &= \delta_2\delta_1 - \alpha_2\alpha_1 - \beta_2\beta_1 - \gamma_2\gamma_1 \\ \alpha_3 &= \alpha_2\delta_1 + \delta_2\alpha_1 + \gamma_2\beta_1 - \beta_2\gamma_1 \\ \beta_3 &= \beta_2\delta_1 - \gamma_2\alpha_1 + \delta_2\beta_1 + \alpha_2\gamma_1 \\ \gamma_3 &= \gamma_2\delta_1 + \beta_2\alpha_1 - \alpha_2\beta_1 + \delta_2\gamma_1. \end{aligned} \tag{20}$$

These are probably more easily remembered in the form

$$q_3 = Q_{q_2} q_1,$$

where q_3 and q_1 are understood to be column vectors like $q = \begin{bmatrix} \delta \\ \alpha \\ \beta \\ \gamma \end{bmatrix}$, and Q_q is defined in (18).

q_3 is called the quaternion product of q_2 and q_1, and is denoted by

$$q_3 = q_2 * q_1.$$

(The sign differences explained below between q and Hamilton's quaternion, $q_H = \bar{q}$, must be borne in mind when comparing (20) with, for example, the quaternion product defined in Whittaker (1904), page 9. Equation (20) corresponds to $\bar{q}_{H_3} = \bar{q}_{H_2} * \bar{q}_{H_1}$.) By comparing the two products, it is seen that generally $q_2 * q_1 \neq q_1 * q_2$, that is, quaternion products are noncommutative. In summary,

$$\begin{aligned} P_{q_2} P_{q_1} &= P_{q_2 * q_1} \\ Q_{q_2} Q_{q_1} &= Q_{q_2 * q_1}. \end{aligned} \tag{21}$$

Whereas the quaternion product does not commute, a directly verifiable property of P and Q defined in (18) is that they *always* commute, that is,

$$P_{q_2}Q_{q_1}=Q_{q_1}P_{q_2}. \tag{22}$$

Using (21) and (22) we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & M_{\hat{q}_2}M_{\hat{q}_1} & \\ 0 & & & \end{bmatrix} = T_{\hat{q}_2}T_{\hat{q}_1}=P_{\hat{q}_2}Q_{\hat{q}_2}P_{\hat{q}_1}Q_{\hat{q}_1}$$
$$=P_{\hat{q}_2}P_{\hat{q}_1}Q_{\hat{q}_2}Q_{\hat{q}_1}=P_{\hat{q}_2*\hat{q}_1}Q_{\hat{q}_2*\hat{q}_1}=T_{\hat{q}_2*\hat{q}_1}$$
$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & M_{\hat{q}_2*\hat{q}_1} & \\ 0 & & & \end{bmatrix},$$

so that

$$M_{\hat{q}_2}M_{\hat{q}_1}=M_{\hat{q}_2*\hat{q}_1}, \tag{23}$$

a result that is difficult (or at least algebraically "messy") to "discover" and prove by any other route.

Since $Q_q^tQ_q=Q_qQ_q^t=\nu^2I=\nu^2Q_I$, where $I=(1, 0, 0, 0)$ and $Q_{q_1}Q_{q_2}=Q_{q_1*q_2}$, we have $\bar{q}*q=q*\bar{q}=\nu^2I$ or $q^{-1}*q=q*q^{-1}=I$ for any q where q^{-1} is defined as $\bar{q}/\nu^2$. For normalized $\hat{q}$

$$\hat{q}^{-1}=\bar{\hat{q}}. \tag{24}$$

Using equation (23) and noting that $M_I=I$, we see that $M_qM_{q^{-1}}=I$. These formulas are helpful in formulating the differential forms for the q parameterizations discussed in the next section.

Since with $\nu^2=\delta^2+\alpha^2+\beta^2+\gamma^2$ as above, $P_{q_1}^tP_{q_1}=\nu_1^2I$ and $P_{q_2}^tP_{q_2}=\nu_2^2I$. We now have, using $P_{q_3}=P_{q_2}P_{q_1}$ in $P_{q_3}^tP_{q_3}=\nu_3^2I$ where $q_3=q_2*q_1$, the equation

$$\nu_3=\nu_2\nu_1. \tag{25}$$

Thus if q_1 and q_2 are normalized ($\nu_1=1$ and $\nu_2=1$), $q_3=q_2*q_1$ is also normalized.

Equations (19b), (20), and (25) are the ones needed for photogrammetric operations. However, it is interesting to digress at this point to discuss the concept of a quaternion, the fundamental and useful properties of which have in fact already been derived above but which are usually approached somewhat differently. While Hamilton's quaternions have a certain intrinsic fascination and historical interest, one hastens to point out that further discussion of quaternions is unnecessary once the above formulas are in hand. Quaternions have the reputation of being obscure, complicated and old fashioned—perhaps deservingly. Their ill-repute probably springs from early attempts to force the quaternion apparatus to do tasks that are now done more simply by vector and matrix methods. The chief interest here is the way in which quaternion-related formulas can be used to parameterize rotations. This is, in the writer's opinion, a use for which quaternions are particularly well suited, whatever other shortcomings they may possess. If we take I_P, E_P, J_P, K_P and I_Q, E_Q, J_Q, K_Q to be P and Q, respectively, for

$$\hat{q}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\-1\end{bmatrix},$$

then it follows that $P_q=\delta I_P-\alpha E_P-\beta J_P-\gamma K_P$, $Q_P=\delta I_Q-\alpha E_Q-\beta J_Q-\gamma K_Q$, and in both groups

$$E^2=J^2=K^2=EJK=-I, EJ=-JE=K,$$
$$JK=-KJ=E, \text{ and } KE=-EK=J.$$

Furthermore, all of I_P, E_P, J_P, and K_P commute with I_Q, E_Q, J_Q, and K_Q. The two systems I, E, J, K have the same multiplication table as $1, i, j, k$ where i, j, k are the so-called hypercomplex units which give a generalization of the idea of a complex number. Then Hamilton's quaternion is the hypercomplex number $d+ai+bj+ck$. In this paper, q denotes the hypercomplex number given by $q=\delta-\alpha i-\beta j-\gamma k$. The minus signs are inserted so that all formulas are consistent from the beginning with an M representing a rotation of the coordinate system, whereas the positive hypercomplex units i, j, k defined by Hamilton are more appropriate for M^t representing either motions of points in a fixed coordinate system or a sign convention for rotation angles opposite to that used here. This procedure is suggested by that of Courant-Hilbert.

Because of the correspondence between $1, i, j, k$ and the basic 4 by 4 matrices I, E, J, K, it follows that the product of q_2 and q_1 produced by multiplying out $(\delta_2-\alpha_2i-\beta_2-\gamma_2k)$ $(\delta_1-\alpha_1i-\beta_1j-\gamma_1k)$ and collecting terms coefficient to $1, -i, -j, -k$ after use of the above multiplication table for $1, i,$

j, k is the same as that in (20) above. Here, it is more expedient to visualize q as a 4-tuple subject to a certain product rule, rather than as a hypercomplex number.

As an example of the use of the hypercomplex numbers to express relations that for most purposes are more readily expressed in matrix-vector notation, the quaternion version of $\bar{x}'=M\bar{x}$, M given in terms of $\hat{q}$ by (19a) is $x' = \hat{q}*x*\bar{\hat{q}}$, where $x=x_1i+x_2j+x_3k$, and $x'=x_1'i+x_2'j+x_3'k$. The writer finds it much easier to use the 4 by 4 or 3 by 3 matrix forms wherever possible rather than hypercomplex numbers even though from a purely mathematical point of view the two are equivalent.

The history of quaternions makes interesting reading. There are numerous anecdotes associated with their development (Newman 1956). Also, see Whittaker (1904), Ames and Murnaghan (1929), and MacMillan (1936) for concise presentation of the basic quaternion formulas derived above.

The geometric significance of the q parameters can be deduced from the 4 by 4's considered above. The first column of $P_qP_q^t=I$ can be written $P_qq=1$ where $q'=(\delta, \alpha, \beta, \gamma)$ and $1'=(1, 0, 0, 0)$. Since $Q_q1=q$, we have $Q_qP_qq=q$ or, since P and Q commute, $T_qq=q$ where $T_q=P_qQ_q$, as before. Recalling the submatrix form of T_q, this gives

$$M\bar{r}=\bar{r}$$

where the vector $\bar{r}=\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$. This implies that $\bar{r}$ must be a vector along the axis of the rotation represented by M, since only these vectors are not changed by the rotation.

The equivalence of every proper orthogonal 3 by 3 matrix to a rotation about an axis is a result known as Euler's Theorem. The axis is designated the Euler axis (not necessarily a coordinate axis), and the angle of rotation about this axis is called the Euler angle (not to be confused with the Eulerian angles mentioned elsewhere). For a detailed proof and discussion, see Goldstein (1950).

To find the significance of δ and of the length of $\bar{r}$ (denote $r=|\bar{r}|=(\alpha^2+\beta^2+\gamma^2)^{1/2}=(1-\delta^2)^{1/2}$ since $\delta^2+r^2=1$), consider the trace (sum of the diagonal elements) of the matrix M, tr M. From (19a) this is $4\delta^2-1$. If we have $\bar{x}'=M\bar{x}$ and $\bar{y}=M_1\bar{x}$, $\bar{y}'=M_1\bar{x}'$, then $\bar{y}'=M_2\bar{y}$ where

$$M_2=M_1MM_1^t. \tag{26}$$

Equation (26) is the transformation formula for M corresponding to a rotation of the coordinates $\bar{x}$ and $\bar{x}'$ by M_1 (note in passing that the quaternion form of this formula readily leads to the above-mentioned equation $x'=\hat{q}*\times *\hat{q}$.)

Furthermore, since tr $AB=$ tr BA, tr $M_2=$ tr $MM_1^tM_1=$ tr M, so that the trace of M is invariant with respect to rotation of coordinates. Specifically, we may consider trace M in a coordinate system in which the Euler axis is the 3 axis so that $M=R_3(\theta)$, θ the Euler angle. Then tr $M=$ tr $R_3(\theta)=1+2\cos\theta$ and by the invariance of the trace,

$$1+2\cos\theta=4\delta^2-1$$

or $\delta=\cos\theta/2$, leading to $r=(1-\delta^2)^{1/2}=\sin\theta/2$. $\hat{r}=\bar{r}/r$, the vector of direction cosines of the Euler axis, can be written

$$\hat{r}=\begin{bmatrix}\cos\phi & \cos\lambda\\ \cos\phi & \sin\lambda\\ \sin\phi & \end{bmatrix}$$

where ϕ, λ are the spherical latitude and longitude of the Euler axis.

In summary,

$$\begin{aligned}\delta&=\cos\frac{\theta}{2}\\ \alpha&=\sin\frac{\theta}{2}\cos\phi\cos\lambda\\ \beta&=\sin\frac{\theta}{2}\cos\phi\sin\lambda\\ \gamma&=\sin\frac{\theta}{2}\sin\phi\end{aligned} \tag{27}$$

Equation (27) establishes the geometric meaning of the q parameters in terms of the direction cosines of the Euler axis and the Euler angle θ. If θ is replaced by $\theta+360°$, q is replaced by $-q$, but the two rotations, insofar as they affect the coordinates $\bar{x}$, are equivalent. This is the geometric sense of the fact, noted above, that q and $-q$ produce the same M.

There are several more interesting geometric interpretations of rotations in terms of spherical triangles or stereographic projections that are closely related to the q parameterization (see Ames and Murnaghan 1929, and Whittaker 1904). Only one additional geometric interpretation will be considered here—one giving a geometric interpretation of the formula $\bar{x}'=M\bar{x}$.

From equation (19b) the following representation of M follows by direct expansion and comparison, using $\delta^2+\alpha^2+\beta^2+\gamma^2=\nu^2$:

$$M=I+\frac{2\delta}{\nu^2}S_{\bar r}+\frac{2}{\nu^2}S_{\bar r}^2,$$

$S_{\bar r}$ a skew-symmetric 3 by 3 with $\bar r=\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$. Or for $\nu^2=1$,

$$M=I+2\delta\, S_{\bar r}+2S_{\bar r}^2. \tag{28a}$$

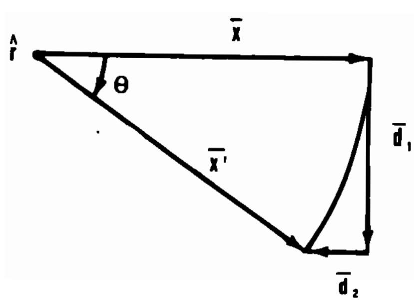
Therefore, using $S_{\bar r}\bar x=-(\bar r\times\bar x)$, $\bar x'=M\bar x$ becomes

$$\bar x'=\bar x-2\delta(\bar r\times\bar x)+2\bar r\times(\bar r\times\bar x). \tag{29}$$

Each term in this equation has a geometric interpretation. Substituting $\delta=\cos\frac{\theta}{2}$, $\bar r=\sin\frac{\theta}{2}\hat r$ produces

$$\bar x'=\bar x-\sin\theta\,(\hat r\times\bar x)+(1-\cos\theta)\,\hat r\times(\hat r\times\bar x).$$

Assume $\bar x$ to be in the plane of the page and $\hat r$ to be out of the page, and consider the vector $\bar x$ to move $\bar x\rightarrow\bar x'$ in a fixed coordinate system so that θ represents clockwise rotation about $\hat r$. Then (29) has the following interpretation



where $\bar x'=\bar x+\bar d_1+\bar d_2$. A similar construction applies when $\hat r$ and $\bar x$ are not perpendicular.

The choice of the sign conventions for θ and $\hat r$ in terms of q were not emphasized in the development to this point. Probably the most convenient way to pin them down "after the fact" is to note that substitution of equation (27) into (19a) for the case $\hat r^t=(1,0,0)$, for example, yields $R_1(\theta)$, and therefore the well-established conventions for the R_i's apply. M_q defined as in equation (19) represents a right-handed screw rotation about the positive Euler axis. Where a choice of either plus or minus is possible (e.g., a square root) in the above development, it has been made to conform with previously defined R_i's.

One additional useful formula can be obtained from the 4 by 4's with ease. Since $P_{\hat q}Q_{\hat q}=T_{\hat q}$ and $Q_{\hat q}^{-1}=Q_{\hat q}^t=Q_{\bar q}$, it follows that $P_{\hat q}=T_{\hat q}Q_{\bar q}$. Expanded in submatrix form, this gives

$$(\delta I+S_r)=M(\delta I-S_r)$$

or

$$M=(\delta I+S_r)(\delta I-S_r)^{-1} \tag{28b}$$

when the inverse exists. Similarly, using $P_{\hat q}^{-1}=P_{\bar q}$, one obtains $M=(\delta I-S_{\bar r})^{-1}(\delta I+S_{\bar r})$. These equations are the basis of an application of the q parameterizations to absolute orientations by Schut (1961). $\bar x'=M\bar x$ becomes $(\delta I-S\bar r)\bar x'=(\delta I+S\bar r)\bar x$, or $\delta(\bar x'-\bar x)=S_{\bar r}(\bar x'+\bar x)$. Letting $\bar s=\frac{1}{\delta}\bar r$ and using $S_{\bar r}\bar y=-S_{\bar y}\bar r$, this becomes $\bar x-\bar x'=S_{\bar x'+\bar x}\bar s$, an equation linear in $\bar s$. From the least-squares solution for $\bar s$ with the residuals on these equations, $\delta^2=1/(1+|\bar s|^2)$ and $\bar r=\delta\bar s$.

Most of the above results are collected under the headings of Cayley's theorem and Hamilton's theorem on quaternion products, which along with Euler's theorem, contain the entire story of the q parameters (see Courant and Hilbert 1937). Attempts to trace the specific results back further than this only result in a proliferation of trivial algebraic variations which obscure the unity of the subject. One motivation of this report has been to emphasize this unity by suggesting that various parameterizations of rotations, for example those surveyed by Schut (1958–59), are in fact very closely related. This is true also of additonal variations, such as the Cayley-Klein parameters, not covered here.

Development of Differential Forms

Differential rotations as discussed in this report are closely related to discussions of angular velocity in, for example, Frazer, Duncan, and Collar (1938). Whittaker (1904) gives a formula for the angular velocity vector in terms of a quaternion parameterization. None tell the entire story however. If one sets $q'=q+\Delta q=Q*q$ where $Q=I+\Delta q*q^{-1}$, then

$$M_{q'}=M_QM_q \tag{29a}$$

or

$$M_{q'}=M_{Q*q} \tag{29b}$$

where, using $q^{-1}=\bar{q}$, $Q=I+\Delta q * q^{-1}$ is explicitly given by

$$\begin{aligned} \delta_Q &= 1+\delta\Delta\delta+\alpha\Delta\alpha+\beta\Delta\beta+\gamma\Delta\gamma \\ \alpha_Q &= -\alpha\Delta\delta+\delta\Delta\alpha+\gamma\Delta\beta-\beta\Delta\gamma \\ \beta_Q &= -\beta\Delta\delta-\gamma\Delta\alpha+\delta\Delta\beta+\alpha\Delta\gamma \\ \gamma_Q &= -\gamma\Delta\delta+\beta\Delta\alpha-\alpha\Delta\beta+\delta\Delta\gamma \end{aligned} \tag{30}$$

Using $\delta^2+\alpha^2+\beta^2+\gamma^2=1$ for q and q' implies that, to the first order in the increments, $\delta\Delta\delta+\alpha\Delta\alpha+\beta\Delta\beta+\gamma\Delta\gamma=0$, so that

$$\Delta\delta=-(\alpha\Delta\alpha+\beta\Delta\beta+\gamma\Delta\gamma)/\delta \tag{31}$$

and

$$\delta_Q=1. \tag{32}$$

Substituting equation (31) into the last three equations of (30) gives

$$\bar{r}_Q=\begin{bmatrix}\alpha_Q\\ \beta_Q\\ \gamma_Q\end{bmatrix}=\frac{1}{\delta}\begin{bmatrix}\delta^2+\alpha^2 & \alpha\beta+\delta\gamma & \alpha\gamma-\delta\beta\\ \alpha\beta-\delta\gamma & \delta^2+\beta^2 & \beta\gamma+\delta\alpha\\ \alpha\gamma+\delta\beta & \beta\gamma-\delta\alpha & \delta^2+\gamma^2\end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta\beta\\ \Delta\gamma\end{bmatrix}. \tag{33}$$

If we define $\bar{\omega}=2\bar{r}_Q$ (for reasons shortly apparent), we have

$$\bar{\omega}=C_Q\begin{bmatrix}\Delta\alpha\\ \Delta\beta\\ \Delta\gamma\end{bmatrix} \tag{34}$$

where C_Q is two times the 3 by 3 coefficient matrix of (33). Compare (34) to equation (13). Also note that

$$\begin{bmatrix}\Delta\alpha\\ \Delta\beta\\ \Delta\gamma\end{bmatrix}=C_Q^{-1}\bar{\omega} \tag{35}$$

where C_Q^{-1} is given by

$$C_Q^{-1}=\frac{1}{2}\begin{bmatrix}\delta & -\gamma & \beta\\ \gamma & \delta & -\alpha\\ -\beta & \alpha & \delta\end{bmatrix},$$

as can be directly confirmed in $C_Q^{-1}C_Q=I$.

If one substitutes Q given by equation (30) into M_Q given by equation (19a) or equation (28a) and discards all terms of second order in Δq, the result is

$$M_Q=I+S_{\bar{\omega}} \tag{36}$$

where

$$\bar{\omega}=2\bar{r}_Q \quad\text{or}\quad \bar{r}_Q=\bar{\omega}/2. \tag{37}$$

Thus the change in M due to the change in q, $\Delta M=M_{q+\Delta q}-M_q=M_{q'}-M_q$, with $M_{q'}=M_QM_q$, is given by

$$\Delta M=S_{\bar{\omega}}M \tag{38}$$

as expected for any parameterization [equation (7b) above].

To compute q' from the least-squares value of $\bar{\omega}$ resulting from the observation equations (6b) produced by use of the differential form (38), there is a choice between $q'=q+\Delta q$, where Δq is obtained from $\bar{\omega}$ by (35) and (31), or $q'=Q*q$, where Q is given by (32) and (37). That is,

$$\begin{aligned} \delta_Q &= 1\\ \alpha_Q &= \omega_1/2\\ \beta_Q &= \omega_2/2\\ \gamma_Q &= \omega_3/2. \end{aligned} \tag{39}$$

Substitution shows the two procedures to be equivalent. Use of $q'=Q*q$ from the beginning, leading directly to equation (38) and equation (39), would have given a derivation in which C_Q and C_Q^{-1} never appeared. They were introduced here to further the analogy with the conventional parameterization and to support the claim made earlier that in the q parameterization, C or C^{-1} need never be actually computed when $q'=Q*q$ is used. A second, no less important reason is that C_Q completes the partial differentiation of x and y with respect to the new parameters, thus showing that the observation equations (6b) are equivalent to those required by the Taylor series, Newton-Gauss formalism with the addition of the exact linear substitution in equation (35).

Two comments can be made about equation (39): (a) Q, based on $\bar{\omega}=2\bar{r}_Q$ from equation (37) and $\delta_Q=1$, is normalized to the first order only. If we do not renormalize q, the resulting M matrix will be a scalar times an orthogonal matrix. To avoid this, Q can be renormalized by $\hat{Q}=Q/\nu$,

$$\nu=(\delta_Q^2+\alpha_Q^2+\beta_Q^2+\gamma_Q^2)^{1/2}$$

or, by equation (24), q' can be renormalized, or, and this is the preferred computational procedure; the renormalization is postponed until the computation of M—equation (19b)—in which case the square root is not needed. Another possibility is that the scale be left in M and q and removed only at the last step of the Newton-Gauss iteration, but for simplicity this is not done here. (b) Equation (39) is not the only way of establishing an equation valid to the first order relating $\bar{\omega}$ and Q; other possibilities result from the retention of some second order terms in Q. Specifically, a natural procedure, one also appropriate for the quadratic iteration mentioned earlier, is to perform first the exact substitution $\bar{\omega} = 2\delta\bar{r}_Q$, obtaining

$$M = I + S_{\bar{\omega}} + (1/l)S_{\bar{\omega}}^2 \tag{40}$$

where $l = 1 + (1 - |\omega|^2)^{1/2}$ and then drop second order terms in $\bar{\omega}$, again giving equation (36). This possibility is mentioned only to point out its relevance to the direct quadratic iteration and to point out that the conventional parameterization and equation (40) both produce equations relating $\bar{\omega}$ and Q which possess singularities at which the Newton-Gauss procedure would break down but that the adopted choice is free of singularities.

Choice of Computational Formulas

Various alternative computational procedures are possible using the above formulas. M can be updated by either (29a) or (29b). $\bar{x}' = M\bar{x}$ can be computed from (a) $M_{q'}\bar{x}$, (b) $M_{\bullet}\bar{x}_{l-1}$ or, (c) the cross product form of equation (29) above. Several combinations are possible. Each combination is examined in detail to determine the number of products needed. This is a matter of counting products, etc., which is not repeated here. The writer's conclusion is that the best procedure is the following: form $M_{q'}$ from $q' = Q{*}q$, renormalize q' during the formation of M, and compute $\bar{x}' = M\bar{x}$ by multiplying M times $\bar{x}$. This yields formulas (9) to (11) above.

Transformation of Parameters

Formulas for q in terms of the Eulerian angles are given in Whittaker (1904, page 11). The following procedure yields formulas for computing q from any given α, β, γ in $R_i(\alpha)R_i(\beta)R_k(\gamma)$ and vice versa. The formulas for the Eulerian angles are a special case of this procedure when $(i, j, k) = (3, 2, 3)$.

Using equation (27) for the elementary rotations $R_l(\theta)$, $l = 1, 2, 3$, q takes the forms,

$$\begin{aligned}
R_1(\theta),\ q_1 &= \left(\cos\frac{\theta}{2},\ \sin\frac{\theta}{2},\ 0\ ,\ 0\right)\\
R_2(\theta),\ q_2 &= \left(\cos\frac{\theta}{2},\ 0\ ,\ \sin\frac{\theta}{2},\ 0\right)\\
R_3(\theta),\ q_3 &= \left(\cos\frac{\theta}{2},\ 0\ ,\ 0\ ,\ \sin\frac{\theta}{2}\right).
\end{aligned} \tag{41}$$

We desire q such that $M_q = R_i(\alpha)R_j(\beta)R_k(\gamma)$. Two successive applications of the quaternion product formula, equation (20), give q either numerically or algebraically. For example, with $(i, j, k) = (3, 2, 1)$, $(\alpha, \beta, \gamma) = (\kappa, \omega, \phi)$, as before, we have $q = q_3{*}q_2{*}q_1$, where q_1, q_2, q_3 are given by equations (41) with $\theta = \phi$, ω, κ, respectively. Thus $q_2{*}\,q_1$, for convenience written in the form $Q_{q_2}q_1$, gives the intermediate result

$$\begin{vmatrix}
\cos\frac{\omega}{2} & 0 & -\sin\frac{\omega}{2} & 0\\
0 & \cos\frac{\omega}{2} & 0 & -\sin\frac{\omega}{2}\\
\sin\frac{\omega}{2} & 0 & \cos\frac{\omega}{2} & 0\\
0 & \sin\frac{\omega}{2} & 0 & \cos\frac{\omega}{2}
\end{vmatrix}
\begin{vmatrix}
\cos\frac{\phi}{2}\\
\sin\frac{\phi}{2}\\
0\\
0
\end{vmatrix}
=
\begin{vmatrix}
\cos\frac{\omega}{2}\cos\frac{\phi}{2}\\
\cos\frac{\omega}{2}\sin\frac{\phi}{2}\\
\sin\frac{\omega}{2}\cos\frac{\phi}{2}\\
\sin\frac{\omega}{2}\sin\frac{\phi}{2}
\end{vmatrix}$$

and similarly $q = q_3{*}(q_2{*}q_1)$ gives the desired formulas

$$\begin{aligned}
\delta &= \cos\frac{\kappa}{2}\cos\frac{\omega}{2}\cos\frac{\phi}{2} - \sin\frac{\kappa}{2}\sin\frac{\omega}{2}\sin\frac{\phi}{2}\\
\alpha &= \cos\frac{\kappa}{2}\cos\frac{\omega}{2}\sin\frac{\phi}{2} + \sin\frac{\kappa}{2}\sin\frac{\omega}{2}\cos\frac{\phi}{2}\\
\beta &= -\sin\frac{\kappa}{2}\cos\frac{\omega}{2}\sin\frac{\phi}{2} + \cos\frac{\kappa}{2}\sin\frac{\omega}{2}\cos\frac{\phi}{2}\\
\gamma &= \sin\frac{\kappa}{2}\cos\frac{\omega}{2}\cos\frac{\phi}{2} + \cos\frac{\kappa}{2}\sin\frac{\omega}{2}\sin\frac{\phi}{2}.
\end{aligned} \tag{42}$$

The same procedure with $(i, j, k) = (3, 2, 3)$ and $(\alpha, \beta, \gamma) = (\psi, \theta, \phi)$ gives, after use of elementary trig identities, q in terms of the Eulerian angles

$$\delta = \cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}$$

$$\alpha = \sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}$$

$$\beta = \sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2}$$

$$\gamma = \cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}$$

which may be compared with Whittaker (1904, page 11).

The most expedient formulas for computing (α, β, γ) in $R_i(\alpha)\ R_j(\beta)\ R_k(\gamma)$ for given (i, j, k) and q are obtained by equating corresponding elements of M expressed in terms of q by equation (19a), and M expressed in terms of $R_i(\alpha)\ R_j(\beta)\ R_k(\gamma)$ by, for example, equation (5). For

$$(i, j, k) = (3, 2, 1), \qquad (\alpha, \beta, \gamma) = (\kappa, \omega, \phi),$$

as above; equating elements of equations (19a) and (5) we have

$$\begin{aligned}
\cos\omega\sin\kappa &= 2(\delta\gamma - \alpha\beta) &&= -m_{21}\\
\cos\omega\cos\kappa &= \delta^2 + \alpha^2 - \beta^2 - \gamma^2 = &&\ \ \ m_{11}\\
\cos\omega\sin\phi &= 2(\delta\alpha - \beta\gamma) &&= -m_{32}\\
\cos\omega\cos\phi &= \delta^2 - \alpha^2 - \beta^2 + \gamma^2 = &&\ \ \ m_{33}\\
\sin\omega &= 2(\alpha\gamma + \delta\beta) &&= \ \ \ m_{31}
\end{aligned}$$

which are easily solved for κ, ω, ϕ. For the Eulerian angles $(i, j, k) = (3, 2, 3)$, $(\alpha, \beta, \gamma) = (\psi, \theta, \phi)$, the corresponding formulas are

$$\begin{aligned}
\sin\theta\sin\phi &= 2(\beta\gamma - \delta\alpha) &&= \ \ \ m_{32}\\
\sin\theta\cos\phi &= 2(\alpha\gamma + \delta\beta) &&= \ \ \ m_{31}\\
\sin\theta\sin\psi &= 2(\beta\gamma + \delta\alpha) &&= \ \ \ m_{23}\\
\sin\theta\cos\psi &= 2(\delta\beta - \alpha\gamma) &&= -m_{13}\\
\cos\theta &= \delta^2 - \alpha^2 - \beta^2 + \gamma^2 = &&\ \ \ m_{33}
\end{aligned}$$

obtained by equating corresponding terms of (19a) and the explicit form of M for the Eulerian angle given, for example, in Whittaker (1904, page 10).

The first and last members of these equations give (ϕ, ω, κ), (ψ, θ, ϕ), etc., in terms of elements of M, should this be needed. It is perhaps worth pointing out that the procedure for computing q from a given M is extremely simple: denote by $S_{\bar{u}}$ the skew-symmetric part of M (any square matrix can be written as the sum of symmetric and skew-symmetric parts, $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$), that is, the middle term of (28a)

$$S_{\bar{u}} = \frac{1}{2}(M - M') = 2\delta S_{\bar{r}}.$$

Then from (19a) or (28a)

$$\bar{u} = \begin{bmatrix} 2\delta\alpha \\ 2\delta\beta \\ 2\delta\gamma \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(m_{23} - m_{32}) \\ \frac{1}{2}(m_{31} - m_{13}) \\ \frac{1}{2}(m_{12} - m_{21}) \end{bmatrix}.$$

Also, from the trace of M, tr $M = m_{11} + m_{22} + m_{33} = 4\delta^2 - 1$, one has $2\delta = (1 + \mathrm{tr}\,M)^{1/2}$, so that $\bar{r} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ is given by $\bar{r} = \frac{1}{2\delta}\bar{u}$.

Differential relations between Δq and $\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$ corresponding to these exact formulas are already easily available from the various C's derived above. However, note that Δq is never needed. The unknowns in the least-squares solution are ω_1, ω_2, ω_3 and Δq is bypassed altogether in the updating of M by formulas (9) through (11). Differential changes in any additional orientation constraints or observations, such as horizon cameras, stellar or sun cameras, and inertial platforms, that can be conveniently expressed in terms of any particular parameterization, are related to ω_1, ω_2, ω_3 by the matrices C and C^{-1}.

Numerical Example

If we take $q = (a, a^3, a^3, a^2)$ where $a = 1/\sqrt{2}$, corresponding to 2θ, ϕ, $\lambda = 45°$, then $\nu^2 = 1$ and

$$M = \frac{1}{4}\begin{bmatrix} 1 & 1 + 2\sqrt{2} & -2 + \sqrt{2} \\ 1 - 2\sqrt{2} & 1 & 2 + \sqrt{2} \\ 2 + \sqrt{2} & -2 + \sqrt{2} & 2 \end{bmatrix}.$$

The orthogonality of M can be directly verified by $M'M = I$. To ten decimal places M and q are

$$M = \begin{bmatrix} .25000\ 00000 & .95710\ 67812 & -.14644\ 66094 \\ -.45710\ 67812 & .25000\ 00000 & .85355\ 33906 \\ .85355\ 33906 & -.14644\ 66094 & .50000\ 00000 \end{bmatrix},$$

$$q = (.70710\ 67812, \quad .35355\ 33906, \quad .35355\ 33906, \quad .50000\ 00000).$$

The three points $\bar{x} = \bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3$ are transformed by this M into $\bar{x}_1, \bar{x}_2, \bar{x}_3$, the columns of M.

Consider the problem of finding the M that transforms $\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3$ into $\bar{x}_1, \bar{x}_2, \bar{x}_3$ all given as above. Generally a least-squares solution is required. This particular problem has an exact solution, which means that the residuals can all be reduced to zero. This problem can be solved in one step by use of $(\delta I - S)^{-1}(\delta I + S)$ as outlined above or otherwise. Here, a Newton-Gauss iteration is used to illustrate the computational formulas of interest. From $\bar{x}' = M\bar{x}$, the linearization yields at each of the three points the three observation equations (at the ith iteration)

$$\bar{v} = M_i\bar{x} - \bar{x}' + S_{\bar{\omega}}M_i\bar{x},$$

or

$$\bar{v} = \bar{x}'_i - \bar{x}' + S_{\bar{\omega}}\bar{x}'_i$$

where

$$\bar{x}'_i = M_i\bar{x}.$$

This can be written as

$$S_{\bar{x}'_i}\bar{\omega} = (\bar{x}'_i - \bar{x}') - \bar{v},$$

leading to normal equations $N\bar{\omega} = U$.

Because of the orthogonality of $\bar{x}$ and therefore of $\bar{x}'_i$ in this particular problem, one finds, for example, by direct expansion, that

$$N = S^t_{\bar{x}'_i}S_{\bar{x}'_i}|_1 + S^t_{\bar{x}'_i}S_{\bar{x}'_i}|_2 + S^t_{\bar{x}'_i}S_{\bar{x}'}|_3 = 2I.$$

Also note that

$$S^t_{\bar{x}'_i}(\bar{x}'_i - \bar{x}') = -S^t_{\bar{x}'_i}\bar{x}' = S^t_{\bar{x}'}\bar{x}'_i,$$

so that

$$U = S^t_{\bar{x}'}\bar{x}'_i|_1 + S^t_{\bar{x}'}\bar{x}'_i|_2 + S^t_{\bar{x}'}\bar{x}'_i|_3.$$

These two results simplify the formation of the normal equations at each stage of the Newton-Gauss iteration and apply to this problem only. Any such simplification is desirable in this illustration, since there is no interest here in the formation and solution of normals, a procedure assumed to be familiar to everyone. The interest is in the rapidity of the convergence of q_i and M_i. The columns of M_i are the computed coordinates $\bar{x}_i = M_i\bar{x}$ at each step. The iteration starts with $q_0 = (1, 0, 0, 0)$, producing $M_0 = I$. U is conveniently formed by writing the numbers in the array form

$$\begin{array}{lll} S_{\bar{x}'} & \bar{x}'_i & \text{point 1} \\ S_{\bar{x}'} & \bar{x}'_i & \text{point 2} \\ S_{\bar{x}'} & \bar{x}'_i & \text{point 3} \end{array}$$

with column-by-column accumulation of products. This array and the resulting normal equations are fully given in table 1 for the first stage only. For subsequent iterations $\bar{\omega}^t_i$, Q_i, q_i, ν^2_i, $\hat{q}_i$, M_i, and $\left(\sum_i \nu^2_i\right)_i$ are given. $\hat{q}_i$ need not be computed, but is given to illustrate the convergence in the parameters. The orthogonality of all M_i holds to within ± 4 in the tenth decimal place.

TABLE 1.—*Example of Newton-Gauss iteration using the* q *parameters*

$$\begin{bmatrix} .0 & .85355\ 33906 & .45710\ 67812 \\ -.85355\ 33906 & .0 & .25000\ 00000 \\ -.45710\ 67812 & -.25000\ 00000 & .0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} .0 & -.14644\ 66094 & -.25000\ 00000 \\ .14644\ 66094 & .0 & .95710\ 67812 \\ .25000\ 00000 & -.95710\ 67812 & .0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} .0 & .50000\ 00000 & -.85355\ 33906 \\ -.50000\ 00000 & .0 & -.14644\ 66094 \\ .85355\ 33906 & .14644\ 66094 & .0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ & 2 & 0 \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1.4142\ 13562 \end{bmatrix}$$

$\bar{\omega}_1^{\ell}$	.5	.5	.70710 67812	
Q_1	1	.25	.25	.35355 33906
q_1	1	.25	.25	.35355 33906
ν_1^2	1.25			
$\hat{q}_1$	.89442 71909	.22360 67977	.22360 67977	.31622 77660

$$M_1 = \begin{bmatrix} .7 & .66568\ 54249 & -.25857\ 86437 \\ -.46568\ 54249 & .7 & .54142\ 13562 \\ .54142\ 13562 & -.25857\ 86437 & .8 \end{bmatrix}$$

$\sum v^2$ .80

$\bar{\omega}_2^{\ell}$	.3	.3	.42426 40687	
Q_2	1	.15	.15	.21213 20343
q_2	.85	.4	.4	.56568 54249
ν_2^2	1.3625			
$\hat{q}_2$	.72819 99927	.34268 23495	.34268 23495	.48462 60262

$$M_2 = \begin{bmatrix} .29541\ 28440 & .94067\ 17225 & -.16693\ 69983 \\ -.47094\ 69520 & .29541\ 28440 & .83122\ 81387 \\ .83122\ 81387 & -.16693\ 69983 & .53027\ 52293 \end{bmatrix}$$

$\sum v_i^2$ .0073394496

$\bar{\omega}_3^{\ell}$	.03027 522956	.03027 522956	.04281 56401	
Q_3	1	.01513 76148	.01513 76148	.02140 78200
q_3	.82577 98164	.41286 69725	.41286 69726	.58388 20719
ν_3^2	1.3637 48853			
$\hat{q}_3$	.70712 64210	.35354 35703	.35354 35704	.49998 61120

$$M_3 = \begin{bmatrix} .25004\ 16631 & .95709\ 28923 & -.14646\ 62489 \\ -.45712\ 06677 & .25004\ 16631 & .85353\ 37497 \\ .85353\ 37497 & -.14646\ 62487 & .50002\ 77753 \end{bmatrix}$$

$\sum v_i^2$ .0000000062

$\bar{\omega}_4^{\ell}$	.00002 77753	.00002 77753	.00003 92804	
Q_4	1	.00001 38876	.00001 38876	.00001 96402
q_4	.82575 68813	.41287 84406	.41287 84406	.58389 82903
ν_4^2	1.363748853			
$\hat{q}_4$	.70710 67814	.35355 33906	.353533906	.50000 00001

$$M_4 = \begin{bmatrix} .25000\ 00002 & .95710\ 67817 & -.14644\ 66097 \\ -.45710\ 67815 & .25000\ 00002 & .85355\ 33910 \\ .85355\ 33909 & -.14644\ 66096 & .50000\ 00003 \end{bmatrix}$$

$\sum v_i^2$ .0000000000

Inspection of $\hat{q}_i$ and M_i shows the convergence to be very fast.

CONCLUSION

In the writer's opinion the added wealth of interpretation of rotations gained by the q parameterization is a point in its favor. However, it should be emphasized that the computational formulas (9) through (11) are all that are needed for application in photogrammetry. These formulas are simple and offer many gains whose accumulative benefit is large. It is hoped that this extended discussion will lead to more use by photogrammetrists of the q parameterization, the basic formulas of which have already been pointed out by Schut (1959), without as yet any noticeable impact on the photogrammetric community as a whole.

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