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On differential scale changes and the satellite Doppler system *z*-shift

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Summary. An often neglected but important role is played by differential scale changes in transforming geodetic datums. A rigorous account of scale variations in any transformation involving reference ellipsoids and its effects on geodetic heights is essential. This role provides a plausible explanation for the reported z-shift between the Doppler defined terrestrial systems and the satellite laser ranging frames.

Key words: geodetic reference system, coordinate transformation, scale, z-shift, satellite geodesy, Doppler

1 Introduction

The repercussions of small differential changes in scale on transforming Cartesian and curvilinear geodetic coordinate systems were briefly discussed in Soler (1976). Pitfalls to be avoided when scale is determined through least-squares adjustments by constraining independently derived baseline lengths were also covered earlier in Leick & van Gelder (1975). Nevertheless, a detailed treatment of the role of scale in geodetic transformation problems seems necessary.

Considering the importance of this topic in relation to the definition and transformation of world and geodetic datums coupled with the expected benefits to be derived by new precise technology and methods such as very long baseline interferometry (VLBI), satellite laser ranging (SLR) and global positioning system (GPS), the necessity of a careful account of appropriate differential scale changes in every coordinate transformation will be stressed. In particular, the consequences of neglecting a rigorous scale correction will be invoked to explain the reported z-shift between the Doppler and SLR coordinate systems.

Although referenced on many occasions, the advantages of the so-called 'Molodenskii equations' (Molodenskii, Eremeev & Yurkina 1960, p. 14) are occasionally misinterpreted. These equations considered only apparent changes in geodetic-network scale due to differential changes in the semimajor axis of the reference ellipsoid. The possible variations in the curvilinear geodetic coordinates as a consequence of a global or space scale change, that is, changes in scale along the axes of the original Cartesian geodetic system, were not treated or mentioned by the Soviet investigators. It was only after observations to artificial near-Earth satellites became available that small differences in scale between conventional datums and/or other global Cartesian systems were detected. A substantial amount of literature was produced at that time covering all types of comparisons between differently realized coordinate systems. As a representative sample, first credit should go to Wolf (1963) for implementing the mathematical concept of 3-D (seven parameters) similarity transformations. On this point see also Veis (1960) and Burša (1966) where transformations involving only shifts and rotations were discussed. Theory was followed by appropriate development of software and actual applications using observed satellite data (e.g. Lambeck 1969; Kumar 1972; Mueller *et al.* 1973) and continued with a series of references covering the subject thoroughly documented in volumes such as Williams & Henriksen (1977), Bomford (1980) and the Proceedings of the First, Second and Third International Symposia on Satellite Doppler Positioning (Proceedings, 1976, 1979, 1982). It appears that some of the transformations treated in these references may not have been performed rigorously. Often the transformation equations are not given explicitly, so hampering detailed comparison.

When transformations between Cartesian and curvilinear geodetic coordinate systems are involved, it is general practice to neglect the contribution that differential changes in scale may have in the redefinition of the ellipsoidal reference surface. This simplification, as will be seen, corrupts the values of the resulting geodetic coordinates, and significant errors based on today's standards of accuracy are introduced in the heights. To clarify these points, the general differential transformation equations will be reviewed first. It principally will establish the basic theoretical framework to be used throughout this paper.

2 General differential transformation equations

Though partially given in some geodetic references, (e.g. Molodenskii *et al.* 1960, p. 14; Hotine 1969, chapter 27; Rapp 1975), the complete non-iterative differential formulas expressing changes in curvilinear geodetic coordinates $(d\lambda, d\phi, dh)$ as a function of differential shifts, rotations, scale and variations (to the second order) in the semimajor axis and flattening of the reference ellipsoid may be written using matrix notation after Soler (1976) as

$$\begin{Bmatrix} (N+h)\cos\phi\, d\lambda \\ (M+h)\, d\phi \\ dh \end{Bmatrix} = \mathbf{R}\left(\begin{Bmatrix} du \\ dv \\ dw \end{Bmatrix}_{7\ \mathrm{par}} + \begin{Bmatrix} du \\ dv \\ dw \end{Bmatrix}_{\delta a, \delta f}\right), \qquad (2.1)$$

where

$$\mathbf{R} = R_1(\tfrac{1}{2}\pi - \phi)R_3(\lambda + \tfrac{1}{2}\pi) = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \qquad (2.2)$$

and

$$\begin{Bmatrix} du \\ dv \\ dw \end{Bmatrix}_{7\ \mathrm{par}} = \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix} + \begin{bmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\epsilon \\ \delta\psi & -\delta\epsilon & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} + \delta s \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{2.3}$$

$$\begin{Bmatrix} du \\ dv \\ dw \end{Bmatrix}_{\delta a,\, \delta f} = -\,[\mathrm{D}] \begin{Bmatrix} \delta a \\ \delta f \end{Bmatrix} - \tfrac{1}{2} \left(\frac{\partial}{\partial a}[\mathrm{D}]\, \delta a + \frac{\partial}{\partial f}[\mathrm{D}]\, \delta f \right) \begin{Bmatrix} \delta a \\ \delta f \end{Bmatrix} \tag{2.4}$$

with

$$[\mathrm{D}] = \begin{bmatrix} \partial u/\partial a & \partial u/\partial f \\ \partial v/\partial a & \partial v/\partial f \\ \partial w/\partial a & \partial w/\partial f \end{bmatrix}. \tag{2.5}$$

Values of the first and second partial derivatives of the Cartesian geodetic coordinates with respect to the ellipsoidal parameters (a, f) in (2.4) are given explicitly in the appendix.

Equations (2.3) and (2.4) are consistent with a transformation from an initial (old) geodetic datum D1 to a final (new) one D2, also denoted symbolically by the convention D2–D1 or the mapping D1→D2. The following nomenclature and signs apply to the differential parameters involved in the transformation

Δx, Δy, Δz = Coordinates of the origin of the Cartesian system (u, v, w) of datum D1 along the frame (x, y, z) of datum D2.

$\delta\epsilon$, $\delta\psi$, $\delta\omega$ = Differential rotations respectively around the axes (u, v, w) of datum D1 to establish parallelism with respect to datum D2. Counter-clockwise rotations as viewed away from the origin are considered positive.

δs = $(s_{\mathrm{D}2} - s_{\mathrm{D}1})/s_{\mathrm{D}1}$. Differential scale change. s denotes the scale known in both datums.

$\delta\bar{a}$ = $a_{\mathrm{D}2} - a_{\mathrm{D}1}$. Change in semimajor axis. See (2.6) below.

δf = $f_{\mathrm{D}2} - f_{\mathrm{D}1}$. Change in flattening.

δa = $\delta\bar{a} + a_{\mathrm{D}1}\delta s = a_{\mathrm{D}2} - (1 - \delta s)a_{\mathrm{D}1}$. Total change in semimajor axis when a differential scale change δs is also involved. (2.6)

The standard notation applies for the rest of the quantities in equations (2.1) through (2.3), namely

$N = a/W$, the principal radius of curvature in the plane of the prime vertical.

$M = a(1 - e^2)/W^3$, the principal radius of curvature in the plane of the meridian,

$W = (1 - e^2 \sin^2 \phi)^{1/2}$,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} (N + h) \cos\phi \cos\lambda \\ (N + h) \cos\phi \sin\lambda \\ [N(1 - e^2) + h] \sin\phi \end{Bmatrix}.$$

Observe that in essense the basic equation (2.1) contains, because of (2.3) and (2.4), what may be considered two different types of scale changes: a global or space scale change represented by δs which primarily affects the unit of length along the three Cartesian axes and an apparent geodetic network scale change influenced by $\delta \bar{a}$. Both changes are possible and compatible. If we know $\delta \bar{a}$ and δs, they can and should be included simultaneously in deterministic equations of the type presented above by redefining the total change in semimajor axis according to equation (2.6). This very point has not been addressed very often, although it may be occasionally of critical importance. It should be emphasized here that a change of δs will not physically modify the size of the reference ellipsoid although the actual magnitude of the semimajor axis will be different because the new basic 'measuring yardstick' has a different unit of length.

On the contrary, a change $\delta \bar{a}$ will leave the scale of 3-D space intact, but the physical size of the ellipsoid defining the datum in question will be modified. Consequently any geodetic quantity related to points on the ellipsoid (ellipsoidal chord distances, geodesics, normal sections, geodetic heights, undulations, etc.) will change in magnitude although the unit of length in which they were measured, that is the scale, remains the same before and after the change of semimajor axis is implemented. Nevertheless, notice that spatial distances between points not on the ellipsoid or physical parameters independent of the reference ellipsoid such as orthometric heights H will remain invariant. Thus, it may be concluded that a $\delta \bar{a}$ change is equivalent to an apparent datum or network scale change although the scale of the space remains constant.

In order to properly understand the differences between the two scaling methods, the specific contribution of each to geodetic heights will be studied in detail in the next section.

3 Comparison of scaling methods and their effects on geodetic heights

From equation (2.1) after neglecting second-order terms and higher of differential quantities, it easily follows that

$$dh = \cos\phi \cos\lambda \Delta x + \cos\phi \sin\lambda \Delta y + \sin\phi \Delta z$$
$$+ Ne^2 \sin\phi \cos\phi (\cos\lambda \delta\psi - \sin\lambda \delta\epsilon) + (aW + h)\delta s - W\delta a + \frac{a(1-f)}{W} \sin^2\phi \delta f. \quad (3.1)$$

Assume that the only parameter involved in the differential transformation between two hypothetical datums is a change in the units of length along the coordinate axes of the first datum, or equivalently, a change δs in the scale of the 3-D space. Then, after substituting in equation (3.1) the values

$$\Delta x = \Delta y = \Delta z = \delta\epsilon = \delta\psi = \delta\omega = \delta\bar{a} = \delta f = 0 \quad (3.2)$$

and making use of (2.6) it is finally concluded, as expected, that the differential change in geodetic height is equal to the original height multiplied by the assumed differential scale factor, that is

$$dh = (aW + h)\delta s - W\delta a = (aW + h)\delta s - W(\delta\bar{a} + a\delta s) = h\delta s \qquad (\delta\bar{a} = 0). \quad (3.3)$$

Consequently, when the scale of the space is changed by δs, every length is multiplied by the total factor $1 + \delta s$. As mentioned in Soler (1976) the correction $dh = h\delta s$ is very small, a few centimetres for the extreme case of heights of about 5 km when $\delta s = 6$ ppm. However, the variation in geodetic (or ellipsoidal) height is significant if we neglect the contribution of global scale-factor changes such as δs (presently accurately determined through space geo-

Table 1. Errors introduced on geodetic latitude and heights when the contributions to the semimajor axis as a consequence of a change in scale δs is neglected.

	$\delta s = 1$ ppm ($\Delta x = \Delta y = \Delta z = \delta\varepsilon = \delta\psi = \delta\omega = \delta\bar{a} = \delta f = 0$)	
ϕ	$d\phi_e = -[Ne^2 \sin\phi\cos\phi/(M+h)10^6]\delta s$	$dh_e = (aW/10^6)\delta s$
0°	0 cm	638 cm
±20°	∓1 cm	638 cm
±45°	∓2 cm	637 cm
±70°	∓1 cm	636 cm
90°	0 cm	636 cm

Note.— If for example: $\delta s = -.827$ppm, then $dh_e(\pm 45°) = -.827 \times 637$cm $= -5.27$m

detic techniques) to the original value of the reference ellipsoid semimajor axis. Moreover, its omission will cause a bias in the heights. The size of this systematic effect is easily computed using the equations just presented. If the contribution of δs to the semimajor axis is ignored, then from (2.6) we have $\delta a = 0$ (still assuming $\delta\bar{a} = 0$ to simplify the reasoning) and equation (3.3) reduces to

$$dh_e = aW\delta s + h\delta s, \tag{3.4}$$

where the subscript e stands for 'error' introduced when the original value of the semimajor axis a is not modified as a consequence of the scale change δs.

Since $h\delta s$ is practically negligible, Table 1 presents the linear errors affecting the geodetic latitudes $d\phi_e$ and heights $dh_e = aW\delta s$ when $\delta s = 1$ ppm. These biases are in practice independent of the value of the selected reference ellipsoid semimajor axis and only change slightly (a few centimetres) with latitude. Due to the rotational property of the reference ellipsoid, a change of δs will not affect the longitude. Notice that a significant bias is added to the geodetic height system when the contribution $a\delta s$ to the semimajor axis of the ellipsoid is ignored.

As we will see next the inclusion of Δr in the often referenced equation

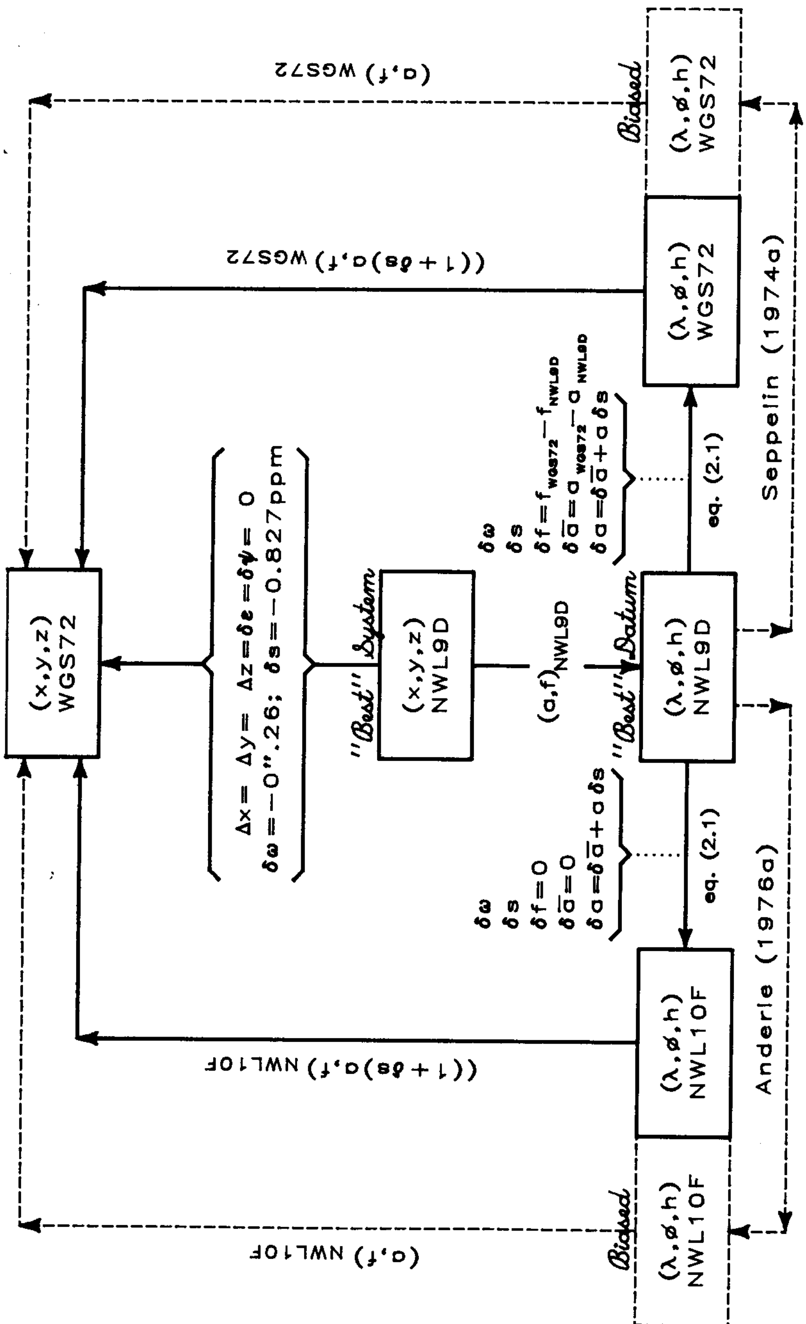
$$dh_s = a \sin^2 \phi\delta f - \delta\bar{a} + \Delta r, \tag{3.5}$$

where

$$\Delta r = -5.27 \text{ m} \equiv dh_e(\pm 45°),$$

introduced by Seppelin (1974a) [due to space limitations this equation is missing from the final published version of the proceedings, consult Seppelin (1974b)] was probably determined empirically. Similar types of equations were later advocated by Anderle (1976a, b). Notice that equation (3.5) is a simplified version of (3.1) besides including the assumptions $\Delta x = \Delta y = \Delta z = \delta\epsilon = \delta\psi = 0$.

Imagine a set of Cartesian coordinates (x, y, z) and a differential change δs of the scale of space. Then, when transforming into curvilinear geodetic coordinates the value of the ellipsoidal semimajor axis must be incremented by the amount $\delta a = a\delta s$; otherwise erroneous answers will result. Remember, that in this circumstance the actual physical size of the



λ (NWL10F) = λ (NWL9D) + 0".26

φ (NWL10F) = φ (NWL9D)

h (NWL10F) = h (NWL9D)−5.27m

NWL9D & NWL10F: a=6378145m; 1/f=298.25

λ (WGS72) = λ (NWL9D) + 0".26

φ (WGS72) = φ (NWL9D) − 0.0232sin2φ

h (WGS72) = h (NWL9D)−0.717sin²φ+4.73m

WGS72: a=6378135m; 1/f=298.26

Figure 1. Schematic diagram of Cartesian and curvilinear coordinate transformation.

ellipsoid remains unaltered! To clarify this point refer to the schematic flow diagram of Fig. 1. By comparing the Doppler satellite systems NWL9D and WGS72 it was determined that small changes in scale and a rotation around the third axis in the sense shown in the figure were required to transform NWL9D → WGS72. Therefore the Cartesian coordinates of the WGS72 system are obtained from the NWL9D through the application of a rotation $\delta\omega$ and the scale factor δs. Because the original NWL9D Cartesian coordinates are assumed to define *a priori* the 'true' ('best') metric space (i.e. its scale), an inverse transformation from $(x, y, z)_{\mathrm{NWL9D}}$ to curvilinear geodetic coordinates $(\lambda, \phi, h)_{\mathrm{NWL9D}}$ using the parameters $(a, f)_{\mathrm{NWL9D}}$ can be implemented without any further consideration.

On the other hand, the transformation from rectangular to curvilinear WGS72 coordinates using $(a, f)_{\mathrm{WGS72}}$ requires a correction of the adopted reference semimajor axis a_{WGS72} by the amount $\delta a = a_{\mathrm{WGS72}}\delta s$ because the semimajor axis of the WGS72 ellipsoid is now measured with the yardstick defined by the true metric space. Valid answers as shown in Fig. 1 are obtained only when this precaution is taken. Seppelin (1974a) probably performed this transformation using the original a_{WGS72}, thus he was forced to change the heights by $\Delta r = -5.27$ m when transforming from $(\lambda, \phi, h)_{\mathrm{NWL9D}}$ to $(\lambda, \phi, h)_{\mathrm{WGS72}}$ in order to arrive at the same results independently of the path followed. Unfortunately, in this instance, the intermediary step, that is the computed values of the geodetic coordinates $(\phi, h)_{\mathrm{WGS72}}$, are incorrect. Incidentally, because $\delta\bar{a}$ in the particular case of transforming curvilinear coordinates from NWL9D to WGS72 is equal to -10 m, the total combined effect introduced on the heights is actually $-W\delta\bar{a} + aW\delta s \cong 4.73$ m (see Fig. 1). The value of $\delta\bar{a}$ naturally will depend on the adopted semimajor axes of the two datums. Thus, ellipsoidal heights which as discussed above should have practically no change, are increased by the amount $\Delta r(\cong aW\delta s)$ only because the correction to the semimajor axis of the reference ellipsoid due to differential scale variation was not taken into account when transforming Cartesian WGS72 into curvilinear geodetic coordinates. Nevertheless, although the intermediate computed values of the curvilinear coordinates $(\lambda, \phi, h)_{\mathrm{WGS72}}$ and $(\lambda, \phi, h)_{\mathrm{NWL10F}}$ are biased and therefore wrong, the final rectangular coordinates $(x, y, z)_{\mathrm{WGS72}}$ obtained from (λ, ϕ, h) using the unscaled semimajor axes are correct. To get an unbiased answer when transforming between curvilinear and rectangular coordinates (or vice versa), the parameters $(a + a\delta s, f)_{\mathrm{WGS72}}$ should have been used because of the differential scale change δs between the assumed 'true' system and the WGS72.

In the following section the direct consequence of these biases on the heights will be explored to possibly explain the detected z-shift between the WGS72 and SLR reference frames.

4 The z-shift of the Doppler-derived satellite system

As remarked previously, the (erroneous) correction dh_e is nearly constant irrespective of the geodetic latitude (it never changes by more than ±1 cm). Hence, a unique systematic error of -5.27 m affecting all geodetic heights was assumed for every simulation investigated here. Since the overall effect on the origin of the reference frame depends fundamentally on the worldwide distribution of points, it is appropriate then to inquire as to what will be the total origin displacement if networks with various station arrangements are considered. By comparing the initial ('true geocentric') set of coordinates with the corrected (biased) ones through a least squares three-parameter similarity transformation, the translation components of the new origin with respect to the true reference system can be determined.

Table 2. Apparent shift of origin in simulated networks caused by an error $dh_e = -5.27$ m in every station due to negligence of the contribution of a scale change $\delta s = -0.827$ ppm to the semimajor axis. Shifts are given in the sense 'true' system to 'biased' system.

Network number	Total	$\phi=-90°$	$\phi=-60°$ $\Delta\lambda=60°$	$\phi=-30°$ $\Delta\lambda=36°$	$\phi=0°$ $\Delta\lambda=30°$	$\phi=30°$ $\Delta\lambda=36°$	$\phi=60°$ $\Delta\lambda=60°$	$\phi=90°$	Δx m	Δy m	Δz m
1	46	1	6	10	12	10	6	1	.0	.0	0.00
2	45	0	6	10	12	10	6	1	.0	.0	-0.12
3	39	0	0	10	12	10	6	1	.0	.0	-0.84
4	29	0	0	0	12	10	6	1	.0	.0	-2.03
5	17	0	0	0	0	10	6	1	.0	.0	-3.47
6	7	0	0	0	0	0	6	1	.0	.0	-4.66
7	1	0	0	0	0	0	0	1	.0	.0	-5.27

Before reporting any results, and in anticipation of more complex verifications using actual Doppler networks, the Δz origin shift dependency in latitude distribution is investigated. Table 2 depicts the results from several imaginary networks with stations symmetrically located around latitude belts over the globe. Note that parallels with stations are sequentially omitted from each solution, one by one, starting from the south pole, in order to identify the contribution of southern sites to the total z-shift.

Intuitively, if we have a network of n points rigorously balanced over the Earth (i.e. the

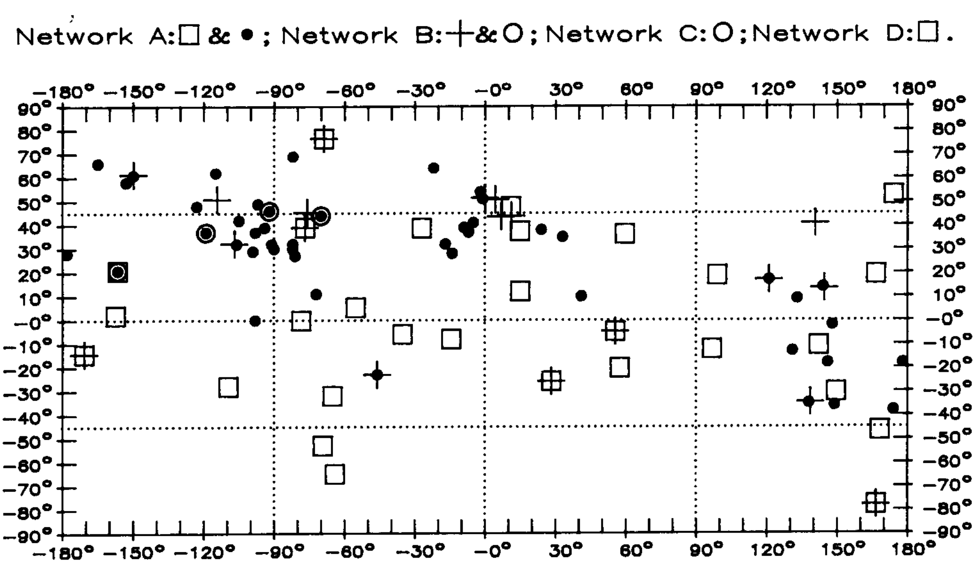


Figure 2. Location of different Doppler stations used in Table 3.

points are symmetrically distributed with respect to the xy-, yz- and zx-planes), in spite of the station heights being erroneously corrected by the constant bias dh_e, no translationary components will result in any direction (i.e. $\Delta x = \Delta y = \Delta z = 0$). This is numerically corroborated by network number 1 of Table 2. As the stations are subsequently removed from this original simulated geometry in a parallel by parallel fashion (starting from the south pole), the Δz origin shift rapidly increases. It reaches its maximum when only one station is left at the north pole. In this situation the apparent Δz-origin shift equals, as expected, the dh_e height error. It is perhaps worth observing that in this last case a change in the station ellipsoidal height will be exactly equal to a change in the corresponding geodetic z-coordinate.

Let us now envisage a transformation in the sense geocentric → Doppler (geodetic) and assume that this single station is erroneously corrected by $dh_e = -5.27$ m; then $z_{\text{Doppler}} - z_{\text{geocentric}} = \Delta z = dh_e = -5.27$ m. The foregoing result implies that the Doppler origin (really the xy-plane) is shifted up by 5.27 m, whereas the ground station (physically attached to the crust) is not affected by translations in the x- or y-directions (i.e. $\Delta x = \Delta y = 0$, $\Delta z = -5.27$ m).

To complement these conclusions a total of four different Doppler networks (see Fig. 2) have been subjected to a similar line of reasoning and computation:

(A) A network of 75 Doppler satellite ground stations providing data for the development of WGS72, as presented in Seppelin (1974a).

(B) The combined TRANET and OPNET networks monitoring at one time the Precise Ephemeris of the Navy Navigation Satellite System (NNSS), as reported e.g. by Anderle (1976a, b). This network consists of 23 (19 + 4) stations.

(C) The OPNET network monitoring the Broadcast Ephemeris. This network consists of four stations, e.g. see Anderle (1976a, b).

(D) A network of 29 stations assumed colocated with the National Oceanic Atmospheric Administration (NOAA) BC-4 network.

Table 3. Apparent origin shifts of several Doppler networks caused by introducing a dh_e error of -5.27 m in all stations, due to negligence of the contribution of a change of scale $\delta s = -0.827$ ppm to the semi-major axis.

NETWORK	Number of stations participating					Shifts Geocentric → Doppler		
	Total	$-90° \le \phi \le -45°$	$-45° < \phi \le 0°$	$0° < \phi \le 45°$	$45° < \phi \le 90°$	Δx	Δy	Δz
A. Doppler/WGS72 [Seppelin, 1974a]	75	4	21	36	14	0.08m	0.78m	-1.30m
B. TRANET/OPNET [Anderle, 1976]	23	1	5	10	7	0.11m	0.59m	-1.90m
C. OPNET [Anderle, 1976]	4	0	0	3	1	1.35m	3.22m	-3.12m
D. BC-4	29	4	12	10	3	-0.31m	0.12m	0.06m

Note. The formal standard error of the translation parameters is equal to the formal standard errors of the station coordinates divided by the number of stations used in the similarity transformation (uncorrelated coordinates were assumed)

Table 3, computed using equal weights for all stations, unequivocally shows the dependence of Δz on the point distribution. Notice that the main influence on the Δz shifts, as also proved by Table 2, is determined primarily by the proportion of sites with $\phi > 45°$ with respect to stations with $\phi < -45°$. The OPNET network, being situated entirely in the United States, also suffers shifts in the x- and y-directions. A significant Δy shift correlates rather well with an assumed error dh_e at the American stations all located in the quadrant between 180°W and 0°.

The tabulated examples of Tables 2 and 3 convincingly illustrate that failure to take rigorous account of scale changes in coordinate transformations involving different reference ellipsoids may cause apparent origin offsets of reference frames, and that these offsets are very much dependent on the stations' geographic pattern.

In our view, the extent to which these consequences are applicable to the z-bias between the Doppler and geocentric reference frames as determined by, e.g. satellite laser ranging, will have to be re-examined by the agencies responsible for the Doppler coordinate solutions. The location of Doppler tracking stations over the Earth and the weight of each individual station in the overall Doppler network adjustment will largely determine the degree to which this z-bias can be cured if errors of the type dh_e are present.

Consequently the geodetic heights of the fundamental network of stations tracking the NNSS satellites and defining the orbital parameters of the precise ephemeris may have been altered in principle by some value dh_e. This bias will translate into a bias in the coordinates of the tracking stations, likewise the computed orbital ephemerides and surely the Doppler or GPS positions resulting from observations to satellites referred to the WGS72 system.

It is conceivable that the initial configuration of the Doppler tracking network generating the ephemerides has been modified or changed over the years for convenience. This may result in small differences in z-shifts, mainly dependent on the range of epochs used when applying the seven parameter similarity transformations between two sets of observing stations, even under the restrictive assumption of a unique network of Doppler receivers. Another possibility postulated by Tscherning & Goad (1985) but deserving a closer scrutiny in years to come, is the correlation between small height variations and solar activity. These changes are not significant enough to substantially affect the origin shifts, although they may influence the Doppler scale when compared with other space systems. Similarly, the full influence of GM on the scale of the Doppler orbit, which in turn may affect the scale imposed on the station coordinates, should be carefully investigated. It should be stressed here that Doppler solutions using different GM constants will result in sets of Cartesian coordinates with different global scale. This in itself, as discussed before, should not greatly affect the heights. Only when transformations between curvilinear coordinates are performed neglecting this GM induced scale change, a height bias will be introduced.

4.1 GRAVIMETRIC ANALYSES AND THE Z-SHIFT

Several investigators have presented different interpretations regarding the geocentricity of the WGS72 systems, despite having followed conceptually the same methodology: comparisons between Doppler and gravimetrically derived geoidal heights. Before attempting to judge the comparative merits and the reasons for the discrepancies of each individual solution, some background information would appear to be pertinent. When undulations from two different systems (for simplicity called old and new) are compared, it is possible to write

$$N_{new} = N_{old} + N_0 + dN, \qquad (4.1)$$

where (Heiskanen & Mortiz 1967, p. 101)

$$\mathrm{N}_0 = (G\delta \mathrm{M}/R\gamma) - (\delta \mathrm{W}/\gamma). \tag{4.2}$$

From the basic relationships $\mathrm{N} = h - H$ (Heiskanen & Moritz 1967, p. 187) it immediately follows (H is a physical quantity and thus ellipsoid independent)

$$d\mathrm{N} = dh = (3.1). \tag{4.3}$$

Substituting the above expression into equation (4.1) we have

$$\mathrm{N}_{new} = \mathrm{N}_{old} + \mathrm{N}_0 + \cos\phi \cos\lambda \Delta x + \cos\phi \sin\lambda \Delta y + \sin\phi \Delta z + \Delta \mathrm{N}, \tag{4.4}$$

where (assuming no rotations, i.e. $\delta\epsilon = \delta\psi = 0$)

$$\Delta \mathrm{N} = (aW + \mathrm{N})\delta s - W\delta a + a(1-f)\sin^2\phi \delta f/W \tag{4.5}$$

N_0 and $\Delta\mathrm{N}$ are (respectively) the physical and geometric contributions to N_{old}, due to differences in mass, potential, scale, semimajor axis and flattening between the two reference ellipsoids to which the old and new undulations refer.

Next, introducing the notation $\bar{\mathrm{N}}_0 = \mathrm{N}_0 + \Delta\mathrm{N}$ we finally arrive at the form preferred by most investigators

$$\mathrm{N}_{new} - \mathrm{N}_{old} = \cos\phi \cos\lambda \Delta x + \cos\phi \sin\lambda \Delta y + \sin\phi \Delta z + \bar{\mathrm{N}}_0. \tag{4.6}$$

The above equation can be used as a mathematical model to compare undulations based on different ellipsoids through a least-squares adjustment where the possible shifts (Δx, Δy, Δz) and $\bar{\mathrm{N}}_0$ may be selected arbitrarily as unknowns. $\bar{\mathrm{N}}_0$ surely will absorb any uncorrected contributions to the undulations not accounted for on the left side of equation (4.6). For example, if on the left side of the equation, uncorrected undulations referring to two ellipsoids differing only in size and shape are used, then after the adjustment, $\bar{\mathrm{N}}_0$ should be nearly equal to the contribution of the quantity $\Delta\mathrm{N} = -W\delta\bar{a} + a(1-f)\sin^2\phi\delta f/W$.

Although this is by no means essential, attention is restricted in the remainder of this section to the case $\delta\mathrm{M} = \delta\mathrm{W} = 0$ (the two ellipsoids defining N_{old} and N_{new} have the same mass and potential). Furthermore, discussion is limited to the long wavelength geoidal information deduced exclusively through a spherical harmonic expansion with potential coefficients belonging to various earth models. No contribution from local surface gravity is implied.

Historically we should commence by mentioning Anderle (1974), who was the first to attempt a comparison between Doppler and gravimetrically derived undulations. His logic, strictly speaking a local spherical approximation, was based on the analysis of differences between two (Doppler and gravimetric) radius vectors at each geoidal point. One vector was obtained by subtracting the mean sea level (MSL) height of the benchmark from its Doppler-derived 'geocentric' distance. The other radius vector was determined by adding the ellipsoidal distance of the station to its undulations as derived from one of the Smithsonian Astrophysical Observatory (SAO) Standard Earth (SE) models, specifically the SE II model. Thus in essence he applied equation (4.6) with $\delta\mathrm{W} = \delta\mathrm{M} = \delta a = \delta f = \delta s = 0$. The explanation of why he found practical geocentricity of the Doppler system is based on the fact that he included only reduced observations from receivers pertaining to the BC-4 network, which is almost symmetrically balanced. This was also corroborated by our own calculations (see Table 3, network D).

Rapp & Rummel (1976) expanded Anderle's idea adding rigor to the technique. Their method became standard and later was emulated by a number of researchers. Essentially they implemented equation (4.6), where N_{old} was obtained using NASA's Goddard (Space Flight Center) earth model number eight (GEM8). N_{new} was computed from Doppler Cartesian coordinates initially scaled by 1 ppm and then converted into curvilinear geodetic coordinates (λ, ϕ, h) using the same ellipsoid as the N_{old} undulations. Thereafter, using $N = h - H$, the Doppler undulations were determined at stations with known orthometric heights. Because all undulations refer now to the same ellipsoid $(\delta \bar{a} = \delta f = 0)$, having the same potential and mass $(\delta W = \delta M = 0)$, the adjusted parameter $\bar{N}_0$ will have no contribution from $N_0 (N_0 = 0)$. Notice, nevertheless, that there still remains $\bar{N}_0 = \Delta N = (aW + N)\delta s - aW\delta s$. Though an attempt was made by Rapp & Rummel to reduce both systems to a common scale [thus the term $(aW + N)\delta s$ was compensated for], the discussed effect of δs on the semimajor axis was not taken into consideration, and therefore inadvertently it remained uncorrected on the left side of equation (4.6). Hence, the resulting adjusted value of $\bar{N}_0$ on the right side of the equation corresponds primarily to the effect of the term $dh_e = aW\delta s$. This is the value obtained by Rapp & Rummel, $\bar{N}_0 \cong -6.36$ m. Again, because they utilized only points belonging to the spatially quasibalanced BC-4 network, no major Δz shift resulted.

The influence of $\bar{N}_0$ on solutions of this type was clarified by the work of Schaab & Groten (1979), where a comparison between geoidal undulations of the GEM8 with other geopotential models was analysed. The main significance of their research was the selection of an exact global distribution of points spaced $10° \times 10°$. When undulations referring to two different earth models (GEM7 and GEM8) defined by the same ellipsoid $(\delta \bar{a} = \delta f = \delta s = \delta W = \delta M = 0)$ were compared, the resulting value of $\bar{N}_0$ and the three shifts were as expected all nearly equal to zero.

A variation of Rapp & Rummel's method was revived by Grappo (1980). He compared Doppler derived undulations obtained by the conventional equation $N = h - H$ with undulations computed from potential coefficients of two of the most advanced earth models at the time: GEM10 and GEM10B. He assumed the same scale, but transformed h (originally derived from the precise $(x, y, z)_{NWL9D}$ coordinates) to the curvilinear WGS72 system by applying Seppelin's equation (3.5), unaware that at this stage he was introducing into the determined Doppler heights the discussed bias of about 5 m. Naturally, this systematic effect on the undulations must be absorbed by some parameters on the right side of equation (4.6). Considering the fact that Grappo did include the term $\bar{N}_0 = -W\delta \bar{a}$ as a parameter, his model really attained the best fitting ellipsoid (with semimajor axis and origin as unknowns) to a set of points miscorrected by dh_e. Conspicuously his results showed little variation in $\delta \bar{a}$ (affected by the distribution of stations as well as the difference in equatorial radius between ellipsoids used) and as anticipated the Δz shift ranging from 4.0 to 5.2 m. Thus, although Grappo's original results and later extensions to other earth models (GEM9, SAO SE III, etc.) reported by Lachapelle & Kouba (1981), appeared to give valid answers, they seemed to be induced by the choice of parameterization and the Seppelin transformation. Observe that if Grappo had used a Doppler network symmetric with respect to the Earth's centre of mass, the value of Δz would not have shown any significant shift and all disagreements would have been confined to the parameter $\delta \bar{a}$. His best balanced Doppler network consisted of 290 points, giving 'a ratio of stations in the two hemispheres of nearly 1 to 1'. Unfortunately, this statement may be misleading because as was pointed out before, the important factor is the ratio of the number of stations with $\phi > 45°$ with respect to $\phi < -45°$. We presume that with 290 stations this ratio was very different from 1 to 1.

Finally, the study in West (1982) should be brought up. The finding of this work was that when geoidal heights based on Seasat altimeter data were compared to gravimetric undulations derived from the DOD WGS72 or GEM10B potential coefficients, a unique

z-shift of about -2.5 m resulted. The decisive element to be accentuated here is that the analysed altimeter data referred to orbits derived exclusively from Doppler observations made by the combined TRANET/Geoceiver Tracking Network. Although two completely independent, earth potential models (DOD WGS72 and GEM10B) were used, the determined z-shift was the same in both instances. This clearly reinforces the hypothesis that the network of Doppler stations tracking the satellite (in this case Seasat) is biased by approximately 2.5 m. Our own simulation shows a bias of -2 m for the TRANET/OPNET network (see Table 3). Earlier Marsh & Williamson (1980), after basically comparing Seasat ephemerides z-values computed at the GSFC with those independently derived at the Naval Surface Weapons Center (NSWC), discovered an unexplained bias of about 4 m. They rightly pointed out the possibility of a z-bias in the tracking stations generating the Seasat Doppler orbits, commenting at the time: 'It must be kept in mind that the origin of this apparent coordinate system error has not been resolved.'

4.2 CORRECTING INDIVIDUAL STATION BIASES

Transformation of curvilinear coordinates neglecting the term $aW\delta s$ when changing geodetic datums with different scales corrupts the geodetic heights by dh_e. There are indications (Anderle 1976b) that some of the Doppler tracking station heights were changed before 1972 and may have been inadvertently biased by a certain amount of dh_e. Because most of these stations are in the northern hemisphere the overall effect will be the introduction of a positive bias in the z-values of the computed ephemerides. As a consequence the resultant z coordinate determined by any Doppler receiver on the surface of the Earth will be also biased and a z-shift appeared on every similarity transformation afterwards.

To remedy this z-bias problem now, and contrary to what intuition may dictate, all the z-coordinates of the Doppler stations should not be corrected by a certain constant value Δz (e.g. the least squares solution of a certain sample of stations). This approach, as will be seen next, does not solve the bias problem and generally distorts the z coordinates (or geodetic heights) of stations in the southern hemisphere.

An error dh_e in the geodetic heights of any arbitrary station i will translate to an error dz_i in the corresponding z-coordinate, equal to

$$dz_i = dh_e \sin \phi_i. \tag{4.7}$$

This is graphically displayed in Fig. 3(a) which shows that the maximum error occurs at the poles, affecting equally geodetic heights or z coordinates. Represented in Fig. 3(b) are the corresponding errors dz for an assumed arbitrary set of points with various geodetic latitudes primarily located in the northern hemisphere. Consequently the least squares (i.e. average) solution of the z-shift will be, as presented in the figure, a line below and parallel to the ϕ axis, resulting in a negative value for Δz as corroborated by experimental evidence.

If the Doppler reference frame is redefined by shifting all stations by the constant amount Δz, then the following residual bias at each station will be left

$$dz_{\phi_i} = dh_e \sin \phi_i + \Delta z. \tag{4.8}$$

Thus, although intuitively it may be thought that a simple Δz correction with opposite sign will take care of any errors introduced by dh_e, this is clearly not the case. Since the Δz shift is dominated by the stations in the northern hemisphere, the applied Δz will largely compensate the dh_e errors only in the northern hemisphere. Because the stations below the equator are a minority, the applied Δz correction will even enlarge the error already introduced by dh_e in the coordinates of the southern stations (in other words the Δz correction goes in the wrong/opposite direction for the southern stations). This is pictorially

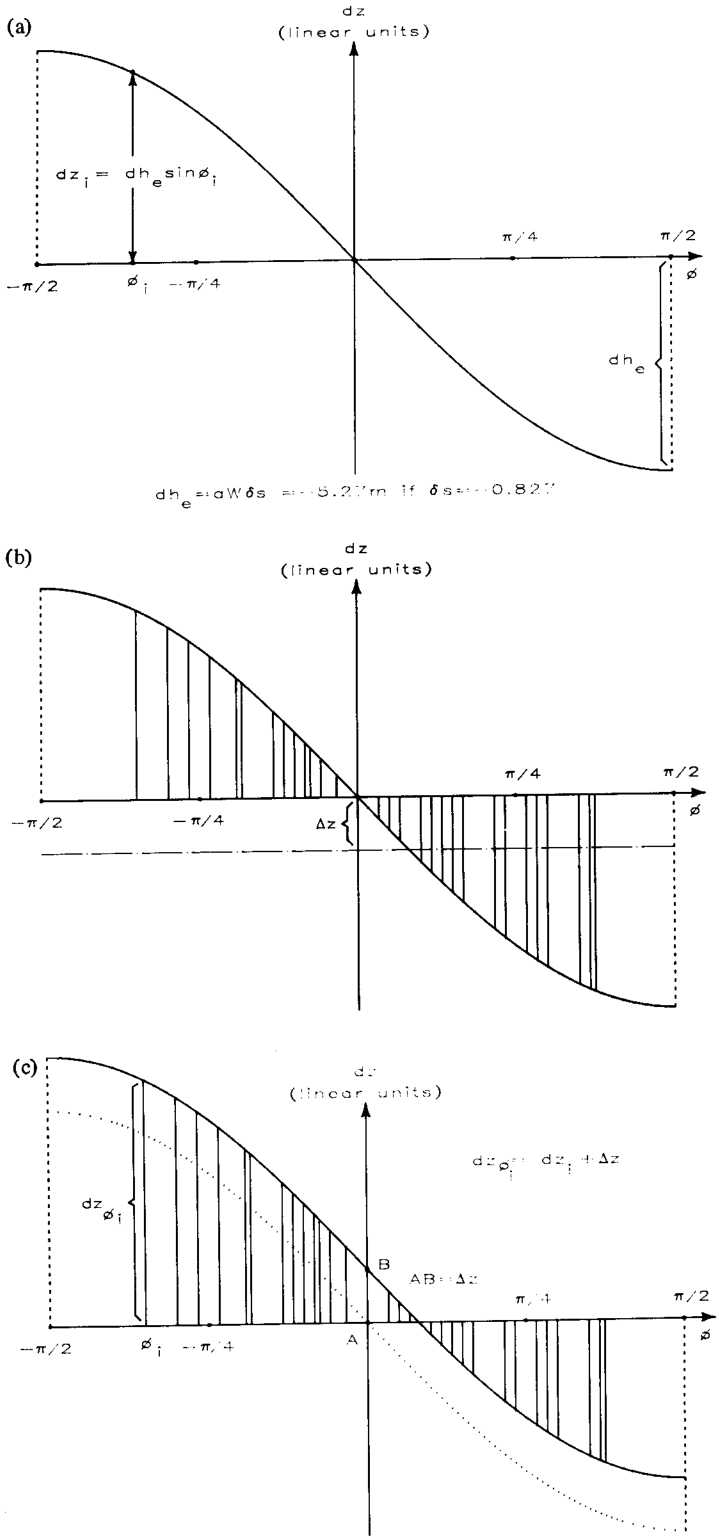


Figure 3. (a) Error dz in stations z-coordinate produced by an error dh_e due to negligence of the correction δs to the semimajor axis. (b) Computed average (least squares) Δz shift from a sample of stations with dh_e errors. (c) Effect of a constant correction Δz in stations z-coordinate.

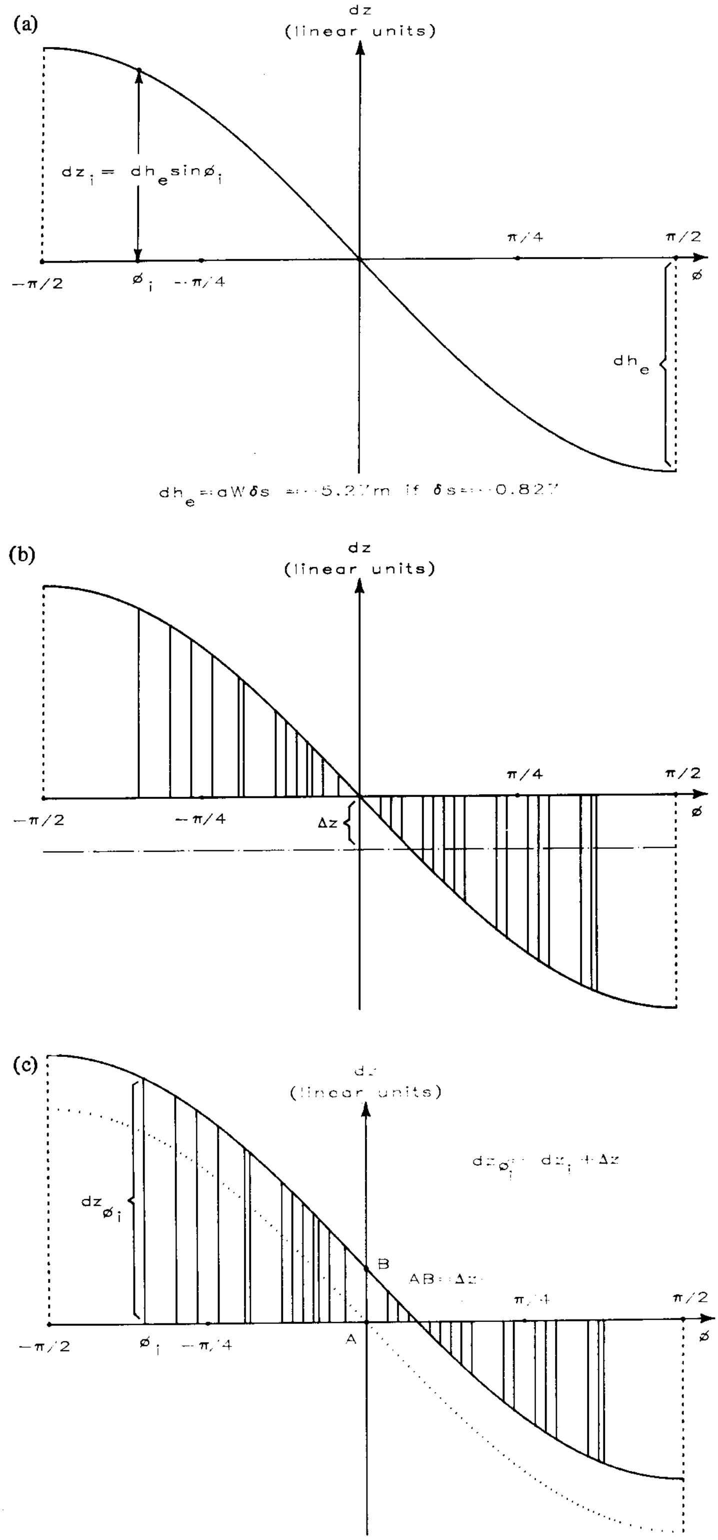


Figure 3. (a) Error dz in stations z-coordinate produced by an error dh_e due to negligence of the correction δs to the semimajor axis. (b) Computed average (least squares) Δz shift from a sample of stations with dh_e errors. (c) Effect of a constant correction Δz in stations z-coordinate.

illustrated in the diagram of Fig. 3(c). Observe that in theory only the stations (if any) along a single parallel of latitude will be rigorously corrected by this method.

The only rational solution should be that the geodetic coordinates of the fundamental network of stations tracking the Doppler satellites and defining the precise ephemerides be corrected to their original values (before their heights were changed). Therefore the actual bias dh_e (a function of the selected scale factor change δs) already applied to the station positions should be known in order to properly correct the coordinates of each individual point using equation (4.7). An alternative approach will be to recompute rigorously (e.g. using mobile SLR) the coordinates of the basic tracking network. With newly generated geocentric ephemerides, the location of a station with respect to the Earth's centre of mass can be determined at any Doppler or GPS receiver site.

5 Conclusions

Among the most crucial problems presently confronting the geodetic and geophysical communities, the proper definition (and materialization) of conventional inertial and terrestrial systems (frames) is singularly important. The impact of accurate geocentricity, scale and orientation of such systems on geodynamic research is fairly obvious. Obstacles to be circumvented when defining and materializing coordinate systems are hinted at by Mueller (1985), who stresses the need for international action in this matter and encourages viable recommendations soon.

Scaling, the main topic of our presentation, deserves supplemental elaboration. When a geodetic reference system (GRS) is defined (see for example Moritz 1984), although this is not explicitly specified, it is none the less implied that the length of the adopted equatorial radius does not refer to an 'ideal' scale unit (e.g. the light-based meter standard) but rather to the best scale which scientists are able to reproduce by means of present geodetic measurements. Scaling methods in geodesy have been greatly improved since the rudimentary French 'toise' was selected by the Lapland and Peru expeditions in the eighteenth century. Recent important advances in time-keeping combined with accurate modern knowledge of the speed of light and the computerized instrumental sophistication of our days have achieved baseline precision beyond the threshold of ordinary applications (Clark *et al.* 1985). Even so, empirical evidence indicates minor discrepancies in scale between different observational techniques (e.g. 0.04 ppm between SLR and VLBI according to Boucher & Altamimi 1985). It is known that values of (x, y, z) geocentric coordinates can be determined using SLR techniques, nevertheless, they must be scaled accordingly, if something other than the SLR inherent scale is adopted as the 'reference scale'. Actually, the Earth's centre of mass is unambiguously known through the results of the SLR observations only after scaling the coordinates in accordance with the adopted global reference scale.

Although VLBI technology and methods, because of their peculiar characteristics – accuracy, reliability, long-term stability and cost-effectiveness (Carter, Robertson & MacKay 1985) may be the best suited at this time to establish the orientation of the conventional inertial and terrestrial systems, the adoption of a global ('universal') scale needed in all geodetic work still requires further studies to conclusively prove which one of the two logical candidates (VLBI and SLR) will reproduce better the 'ideal' unit of length. Some concerns about scale differences between SLR and GLBI solutions were alluded to in Tapley, Schutz & Eanes (1985) where several unconfirmed possible causes were enumerated, although Kolenkiewicz, Ryan & Torrence (1985) did not find any significant scale difference between VLBI and SLR baseline solutions.

In summary, though proper adoption of a geodetic global scale may still be a premature enterprise, nevertheless when transforming existence geodetic datums, since generally their scales will differ, the correction $a\delta s$ to the semimajor axis, amply examined in this paper should be applied consistently. It is vital to realize that current transformations of curvilinear geodetic systems with different scales are not always performed appropriately and this practice may still continue undetected.

Finally, we would like to conclude by emphasizing the immediate priority and attention that should be given to the selection and adoption of a 'global reference scale' derived by any convenient space technique whenever realization of terrestrial reference frames are discussed. Only then can a complete interrelation between different geodetic and/or geophysical products, inferred from various methods (GPS, SLR, VLBI), be rigorously ascertained.

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References

Anderle, R. J., 1974. Transformation of terrestrial survey data to Doppler satellite data, *J. geophys. Res.*, 79, 5319–5333.

Anderle, R. J., 1976a. Error model for geodetic positions derived from Doppler satellite observations, *Bull. Geod.*, **50**, 43–77.

Anderle, R. J., 1976b. Point positioning concept using precise ephemeris, *Proc. Int. Geodetic Symp. Satellite Doppler Positioning*, pp. 47–75, Defense Mapping Agency, Washington, DC.

Bomford, G., 1980. *Geodesy*, 4th edn, Clarendon Press, Oxford.

Boucher, C. & Altamimi, Z., 1985. Towards an improved realization of the BIH terrestrial frame, *Proc. Int. Conf. Earth Rotation Terr. Ref. Frame*, pp. 551–564. Department of Geodetic Science and Surveying, Ohio State University, Columbus.

Burša, M., 1966. Fundamentals of the theory of geometric satellite geodesy, *Trav. Inst. Géophys. Acad. Tchéc. Sci.*, **241**, Československá Akademie Věd, Prague.

Carter, W. E., Robertson, D. S. & McKay, J. R., 1985. Geodetic radio interferometric surveying: applications and results, *J. geophys. Res.*, **90**, 4577–4587.

Clark, T. A., Corey, B. E., Davis, J. L., Elgered, G., Herring, T. A., Hinteregger, H. F., Knight, C. A., Levine, J. I., Lundqvist, G., Ma, C., Nesman, E. F., Phillips, R. B., Rogers, A. E. E., Ronnang, B. O., Ryan, J. W., Schupler, B. R., Shaffer, D. B., Shapiro, I. I., Vanderberg, N. R., Webber, J. C. & Whitney, A. R., 1985. Precision geodesy using the Mark-III Very-Long-Baseline Interferometer system, *IEEE Trans. Geosci. Remote Sensing*, **GE-23**, 438–449.

Grappo, G. A., 1980. Determination of the earth's mean equatorial radius and center of mass from Doppler-derived and gravimetric geoid heights, *Manuscr. Geod.*, 5, 201–216.

Heiskanen, W. A. & Moritz, H., 1967. *Physical Geodesy*, Freeman, San Francisco.

Hotine, M., 1969. *Mathematical Geodesy; ESSA Mongraph No. 2*, US Department of Commerce, National Oceanic and Atmospheric Administration, Rockville, MD.

Kolenkiewicz, R., Ryan, J. & Torrence, M. H., 1985. A comparison between LAGEOS Laser Ranging and Very Long Baseline Interferometry determined baseline lengths, *J. geophys. Res.*, **90**, 9265–9274.

Kumar, M., 1972. Coordinate transformation by minimizing correlations between parameters, *Rep. Dept. Geod. Sci.*, **184**, Ohio State University, Columbus.

Lachapelle, G. & Kouba, J., 1981. Relationship between terrestrial and satellite Doppler systems, in *Reference Coordinate Systems for Earth Dynamics*, eds Gaposchkin, E. M. & Kołaczek, B., pp. 195–203, Reidel, Dordrecht, Holland.

Lambeck, K., 1969. New estimates for the relation of the North American datum to a geocentric satellite reference system, *Studia geophys. Geod.*, **13**, 482–485.

Leick, A. & van Gelder, B. H. W., 1975. On similarity transformations and geodetic network distortions based on Doppler satellite observations, *Rep. Dept. Geod. Sci.*, **235**, Ohio State University, Columbus.

Marsh, J. M. & Williamson, R. G., 1980. Precision orbit analyses in support of the Seasat altimeter experiment, *J. astronaut. Sci.*, **28**, 345–369.

Molodenskii, M. S., Eremeev, V. F. & Yurkina, M. I., 1960. *Methods for Study of the External Gravitational Field and Figure of the Earth*, Translation from Russian (1962), National Technical Information Service, Springfield, VA.

Moritz, H., 1984. Geodetic reference system 1980, *Bull. Geod.*, **58**, 388–398.

Mueller, I. I., 1985. Reference coordinate systems and frames: concepts and realization, *Bull. Geod.*, **59**, 181–188.

Mueller, I. I., Kumar, M., Reilly, J. P., Saxena, N. & Soler, T., 1973. Global satellite triangulation and trilateration for the National Geodetic Satellite Program (Solutions WN12, 14 and 16), *Rep. Dept. Geod. Sci.*, **199**, Ohio State University, Columbus.

Proceedings, 1976. *Proc. Int. Geodetic Symp. Satellite Doppler Positioning*, vols 1 and 2, Defense Mapping Agency, Washington, DC.

Proceedings, 1979. *Proc. 2nd Int. Geodetic Symp. Satellite Doppler Prositioning*, vols 1 and 2, Defense Mapping Agency, Washington, DC.

Proceedings, 1982. *Proc. 3rd Int. Geodetic Symp. Satellite Doppler Positioning*, vols 1 and 2, Defense Mapping Agency, Washington, DC.

Rapp, R. H., 1975. *Geometric Geodesy*, vols I and II, Lecture Notes, published by Dept. of Geod. Sci., Ohio State University, Columbus.

Rapp, R. H. & Rummel, R., 1976. Comparison of Doppler derived undulations with gravimetric undulations considering the zero-order undulations of the geoid, in *Proc. Int. Geodetic Symp. Satellite Doppler Positioning*, pp. 389–397. Defense Mapping Agency, Washington, DC.

Schaab, H. & Groten, E., 1979. Comparison of geocentric origins of global systems from uniformly distributed data, *Bull. Geod.*, **53**, 11–17.

Seppelin, T. O., 1974a. *The Department of Defense World Geodetic System 1972*, paper presented at International Symposium on Problems Related to the Redefinition of North American Geodetic Networks, University of New Brunswick, Fredericton, New Brunswick, Canada, May 20–25.

Seppelin, T. O., 1974b. The Department of Defense World Geodetic System 1972, *Canadian Surveyor*, **28**, 496–506.

Soler, T., 1976. On differential transformations between Cartesian and curvilinear (geodetic) coordinates, *Rep. Dept. Geod. Sci.*, **236**, Ohio State University, Columbus.

Tapley, B. D., Schutz, B. E. & Eanes, R. J., 1985. Station coordinates, baselines, and Earth rotation from LAGEOS Laser Ranging: 1976–1984, *J. geophys. Res.*, **90**, 9235–9248.

Tscherning, C. C. & Goad, C. C., 1985. Correlation between time dependent variations of Doppler-determined height and sunspot numbers, *J. geophys. Res.*, **90**, 4589–4596.

Veis, G., 1960. Geodetic uses of artificial satellites, *Smithson. Contr. Astrophys.*, **3**, 95–161.

West, G. B., 1982. Mean earth ellipsoid determined from SEASAT 1 altimeter observations, *J. geophys. Res.*, **87**, 5538–5540.

Williams, F. L. & Henriksen, S. W. (eds), 1977. *National Geodetic Satellite Program, Parts I and II, NASA SP-365*, National Aeronautics and Space Administration, Washington, DC.

Wolf, H., 1963. Geometric connection and re-orientation of three-dimensional triangulation nets, *Bull. Geod.*, **68**, 165–169.

Appendix A. First and second order partial derivatives of the Cartesian geodetic coordinates (u, v, w) with respect to the ellipsoidal parameters (a, f)

Although the values of the first order partial derivatives are available elsewhere in the literature (e.g. Rapp 1975; Soler 1976), they are included in this paper primarily to make it self-contained. Nevertheless, the expansion to the second order partials and their analytical values are explicitly given here to the authors' knowledge for the first time.

(a) Elements of the matrix [D] in equations (2.4) and (2.5).

$\partial u/\partial a = \cos\phi \cos\lambda/W$ $\quad$ $\partial u/\partial f = a(1-f)\sin^2\phi\cos\phi\cos\lambda/W^3$

$\partial v/\partial a = \cos\phi \sin\lambda/W$ $\quad$ $\partial v/\partial f = a(1-f)\sin^2\phi\cos\phi\sin\lambda/W^3$

$\partial w/\partial a = (1-e^2)\sin\phi/W$ $\quad$ $\partial w/\partial f = (M\sin^2\phi - 2N)(1-f)\sin\phi$

(b) Elements of the matrices $\dfrac{\partial}{\partial a}[\mathbf{D}]$ and $\dfrac{\partial}{\partial f}[\mathbf{D}]$ in equation (2.4)

$$\partial^2 u/\partial a^2 = \partial^2 v/\partial a^2 = \partial^2 w/\partial a^2 = 0$$

$$\partial^2 u/\partial a\,\partial f = \partial^2 u/\partial f\,\partial a = (1-f)\sin^2\phi\cos\phi\cos\lambda/W^3$$

$$\partial^2 v/\partial a\,\partial f = \partial^2 v/\partial f\,\partial a = (1-f)\sin^2\phi\cos\phi\sin\lambda/W^3$$

$$\partial^2 w/\partial a\,\partial f = \partial^2 w/\partial f\,\partial a = (1-f)\sin\phi\,[(1-f)^2\sin^2\phi - 2W^2]/W^3$$

$$\partial^2 u/\partial f^2 = (3M\sin^2\phi - N)\sin^2\phi\cos\phi\cos\lambda/W^2$$

$$\partial^2 v/\partial f^2 = (3M\sin^2\phi - N)\sin^2\phi\cos\phi\sin\lambda/W^2$$

$$\partial^2 w/\partial f^2 = \sin\phi[(1-f)^2\sin^2\phi(3M\sin^2\phi - 4N) - W^2(M\sin^2\phi - 2N)]/W^2$$