Alternative Geometric Determination of Altazimuthal-Distance Covariance Matrices

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Abstract: Conventional equations for determining the variance-covariance matrix of vertical angles, geodetic azimuths and distances are based on the standard law of propagation of covariance using the Jacobian matrix of the corresponding functional relationships. A conceptually simpler geometric approach exclusively dependent on the notion of rotation matrices is presented here. The method completely avoids the cumbersome requirement of taking partial derivatives of non-linear expressions. As an added advantage the method contributes to clarify several points related to dimensional transformations between linear and angular units.

Introduction

The classical geodetic and surveying techniques for orienting networks, charts, and engineering projects when known azimuth reference marks are unavailable, are the determination of astronomic azimuths from observations to stars (Nassau 1948; Roelofs 1950; Mueller 1969). Moreover, in order to account for the orientation degree of freedom, a minimum of one azimuth is required when least squares adjustments are implemented to recover the two dimensional point coordinates from angle (or direction) observations. Otherwise the normal equation matrix **N** is singular, and only inner constraint solutions (based on the pseudoinverse of **N**) are feasible (Pope 1971; Meissl 1982).

With recent advances in satellite geodesy, especially the introduction of new methods and technology such as the Global Positioning System (GPS), the possibility of measuring accurate geodetic azimuths independent of the restrictive constraints imposed by astronomic observations is increasingly apparent (Soler, et al. 1986). Consequently, future use of stars as known reference points in geodetic astronomy may be limited to the determination of first order astronomic latitude and longitude necessary when accurate knowledge of deflections of the vertical is needed.

Several authors have discussed the theoretical problem concerning the computation of the vertical angle (or its complement, geodetic zenith distance), geodetic azimuth, and the spatial distance between two points (standpoint and forepoint) of known geocentric Cartesian coordinates. Final equations derived through different methodologies exist (Wolf 1963; Arnold 1964; Sigl 1969). More lengthy expressions, in terms of the two-

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Note.—Discussion open until November 1, 1987. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on July 30, 1986. This paper is part of the *Journal of Surveying Engineering*, Vol. 113, No. 2, June, 1987. ©ASCE, ISSN 0733-9453/87/0002-0057/$01.00. Paper No. 21555.

point curvilinear geodetic coordinates ($\lambda_i, \phi_i, h_i : i = 1, 2$), are also available in the geodetic literature (Molodenskii, et al. 1960). Different applications of previously mentioned equations using data processed from optical observations to artificial earth satellites have also been discussed (Halmos and Szádeczky-Kardoss 1974; Kaniuth and Zernecke 1979).

It is well known that the variance-covariance matrix of the vertical angle, azimuth, and the distance is readily available by applying the conventional covariance propagation law in terms of the corresponding Jacobian matrices (e.g., Uotila 1967; Mikhail, 1976).

Accurate determination of azimuths, vertical angles, and distances is important in any geodetic surveying work—especially in jobs related to precise engineering projects (e.g., dam deformation, alignment of particle accelerators used in atomic physics and stress-strain analyses used in tetonic and earthquake studies. An alternative, simpler approach to determining corresponding variance-covariance matrices will be presented. The method introduced is appropriate for post-processing, three-dimensional data such as GPS results. The procedure is primarily based on geometric principles and is independent of the more abstract concept of Jacobian matrices; consequently, it is easier to implement when writing algorithms for coding computer programs.

Covariance Matrices Involving Cartesian Coordinates

Important transformations between different local Cartesian coordinate systems generally used when GPS observations are post-processed and analyzed, including interconnecting commutative diagrams, were reviewed in Soler and Chin (1985). The main software output of any GPS reduction program (e.g., Remondi 1984; Goad 1985) is two sets of parameters: the components ($\Delta x, \Delta y, \Delta z$) of the base line vector from point A to B (see Fig. 1) given in the local WGS72 frame (x, y, z) located at A and the corresponding variance-covariance matrix $\Sigma_{(\Delta x, \Delta y, \Delta z)_B}$ at B in the same reference frame. The base (fixed) station (A) is assumed to be known (i.e., we are strictly concerned with the relative position of point B with

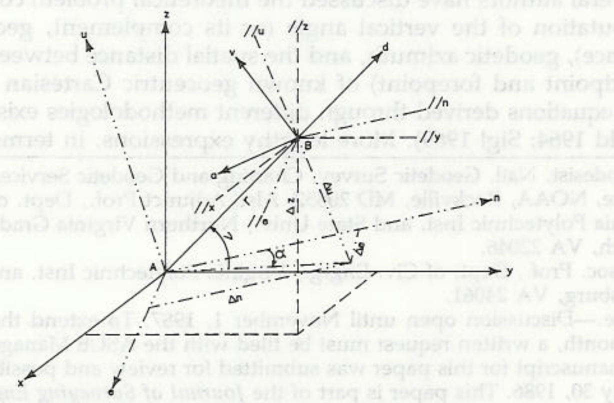

FIG. 1.—Local Cartesian Frames at Standpoint A and Forepoint B

respect to A), and that all local WGS72 frames, whatever their location on the earth's surface, are assumed parallel to the geocentric terrestrial (earth fixed) WGS72 system. For better practical visualization, the $(\Delta x, \Delta y, \Delta z)$ components are usually transformed into the local geodetic frame (e, n, u) [i.e., east, north, up (geodetic zenith)] at A, namely $(\Delta e, \Delta n, \Delta u)$. By its own definition, the local geodetic frame (e, n, u) changes its spatial orientation from point to point and differs only by the local deflections of the vertical from the "true" astronomical horizon system. Adhering to historical precedent (Wolf 1963), only right-handed Cartesian coordinate systems will be used; therefore, departing from the left-handed notation (N:north, E:east, U:up), preferred by some authors (e.g., Heiskanen and Moritz 1967; Rapp 1975; Vaníček and Krakiwsky 1982). In the latter case, transformation matrices between geocentric and local coordinate systems are not proper rotations, and extra permutation (i.e., symmetry or reflection) matrices are involved.

Several frames important to the discussions that follow are depicted in Fig. 1. At point A, the local WGS72 and the local geodetic frames are shown. In general, these frames are related through the mapping

$$\text{R}: (x, y, z) \rightarrow (e, n, u) \qquad (1)$$

where R is the following proper rotation ($R^t = R^{-1}$ and $|R| = +1$) matrix

$$\text{R} = \text{R}_1\left(\frac{1}{2}\pi - \phi\right)\text{R}_3\left(\lambda + \frac{1}{2}\pi\right)$$

$$= \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \qquad (2)$$

and λ, ϕ are the geodetic longitude and latitude, respectively, of the point, referred to the "WGS72 datum."

The transformation between two sets of components of any arbitrary vector expressed in the above systems may be written:

$$\text{R}: (\Delta x, \Delta y, \Delta z) \rightarrow (\Delta e, \Delta n, \Delta u) \qquad (3)$$

$$\text{or:} \quad \begin{Bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{Bmatrix} = \text{R} \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta x \end{Bmatrix} \qquad (4)$$

At point B (Fig. 1), three different reference frames are drawn: one parallel to the local geodetic frame at A; the local WGS72 frame at B (which is parallel to the local WGS72 at A); and a local spherical frame (the reference sphere having origin at A, and radius r equal to the distance AB) denoted by (v, a, d), and pointing in the direction of positive ν (vertical angle of B above the geodetic horizon of A), α (geodetic azimuth), and r (spatial distance between A and B), respectively. This triad is also depicted in Fig. 2, where the relationship between the (e, n, u) and (v, a, d) frames through the polar curvilinear parameters ν and α is shown.

Consider now the mapping

$$\text{R}: (e, n, u) \rightarrow (v, a, d) \qquad (5)$$

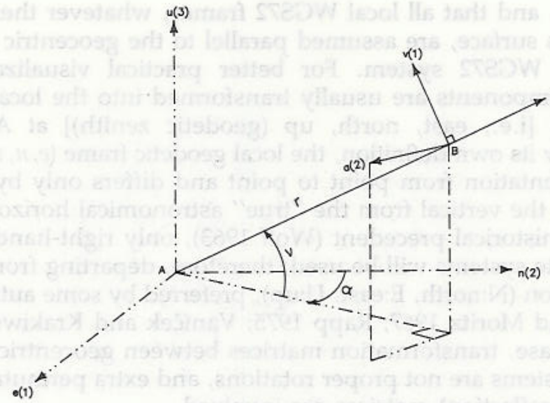

FIG. 2.—Vertical Angle, Geodetic Azimuth, and Distance in Local Geodetic Frame

relating the local geodetic frame at A and the local spherical frame (v, a, d) located at B. This is established (see Fig. 2) through the rotation matrix:

$$\mathrm{R} = \mathrm{R}_3(\pi)\mathrm{R}_2\left(\frac{1}{2}\pi - \nu\right)\mathrm{R}_3\left(\frac{1}{2}\pi - \alpha\right)$$

$$= \begin{bmatrix} -\sin\nu\sin\alpha & -\sin\nu\cos\alpha & \cos\nu \\ \cos\alpha & -\sin\alpha & 0 \\ \cos\nu\sin\alpha & \cos\nu\cos\alpha & \sin\nu \end{bmatrix} \quad \text{(6)}$$

$$\text{where} \quad \nu = \tan^{-1}\left[\frac{\Delta u}{(\Delta e^2 + \Delta n^2)^{1/2}}\right] \quad (7a)$$

$$\alpha = \tan^{-1}\left(\frac{\Delta e}{\Delta n}\right) \quad (7b)$$

$$r = (\Delta e^2 + \Delta n^2 + \Delta u^2)^{1/2} \quad (7c)$$

The following problem can be stated: determine (in linear units) the variance-covariance matrix $\Sigma_{(v,a,d)}$ of the vertical angle, azimuth, and distance between points A and B, given the position of B, with respect to A, $(\Delta x, \Delta y, \Delta z)$ and its covariance matrix $\Sigma_{(\Delta x, \Delta y, \Delta z)_B} = \Sigma_{(\Delta x, \Delta y, \Delta z)}$.

It is well known (e.g., Meissl 1984), that if $\Sigma_{(\Delta x, \Delta y, \Delta z)}$ is diagonalized, the resulting diagonal elements are the eigenvalues (principal or uncorrelated variances) of the matrix. The principal rms errors σ_{x_p}, σ_{y_p}, and σ_{z_p} correspond geometrically to the magnitudes of the three semi-axes of an ellipsoid (standard or mean error ellipsoid) centered at B. The three eigenvectors of $\Sigma_{(\Delta x, \Delta y, \Delta z)}$ determine the orientation of the ellipsoid principal axes x_p, y_p, z_p with respect to the WGS72 local system at B. Readers prefering to avoid completely the eigen theory may consult Wolf (1975), where explicit equations to determine the magnitude and orientation of the error ellipsoid are presented. Recently, Hein and Landau (1983), adapted this formulation, complementing their results with 3D error ellipsoid plots.

Also known (see Appendix I), is the fact that the so-called "pedal surface" (i.e., locus of points with rms σ_γ along the line of arbitrary direction cosines γ_1, γ_2, γ_3) is a surface circumscribed to the error ellipsoid and tangent at the points of intersection with the mean ellipsoid principal axes. Some interesting properties of two-dimensional pedal curves (i.e., Fußpunktskurve = foot point curve), which are sections of the pedal surface along planes containing any of the three principal axes, may be consulted in Köhr (1969), Groten (1969), and Veress (1974). The concept of pedal surface is quite general and applies equally to moments of inertia as well as strain (or stress) problems.

Applying the mapping in Eq. 1 to point B, the relationship between the local WGS72 and geodetic frames at B may be written:

$$R_B : (x,y,z) \rightarrow (e,n,u)_B \quad \ldots\ldots (8)$$

where the rotation matrix $\mathbf{R}_B$ is given by Eq. 2 after replacing λ and ϕ by λ_B and ϕ_B respectively. Consequently, the covariance matrix in the local geodetic frame (e,n,u) at B after applying the standard covariance law is:

$$\sum_{(\Delta e,\Delta n,\Delta u)_B} = R_B \sum_{(\Delta x,\Delta y,\Delta z)_B} R_B^t \quad \ldots\ldots (9)$$

where t denotes transpose.

Mappings such as Eqs. 1, 5 and 8 can be combined in a single commutative diagram as:

$$\begin{array}{ccc} & R_A & \\ (x,y,z) & \rightarrow & (e,n,u)_A \\ R_B \downarrow & \searrow R & \downarrow \mathbf{R} \\ (e,n,u)_B & \rightarrow & (v,a,d)_B \\ & R & \end{array} \quad \ldots\ldots (10)$$

Recall that the $(x,y,z)_{WGS72}$ local frames are always parallel; thus, when only rotations are involved, it is not necessary to identify them by their location. The following equivalent notations are implied $(x,y,z)_A \equiv (x,y,z)_B \equiv (x,y,z)$.

Therefore, the previously stated problem involves the mapping:

$$R: (x,y,z) \rightarrow (v,a,d)_B \quad \ldots\ldots (11)$$

which according to the commutative diagram in Eq. 10 can be solved using the known matrices $\mathbf{R_A}$ (i.e., the matrix R applied to the coordinates of point A) and $\mathbf{R}$. The transformation between the local geodetic system at B and the frame $(v,a,d)_B$ can be materialized through the rotation R also shown in the commutative diagram.

By simple application of the covariance law and the commutative diagram in Eq. 10 we may write:

$$\sum_{(v,a,d)} = R \sum_{(\Delta x,\Delta y,\Delta z)} R^t = \mathbf{R}\, R_A \sum_{(\Delta x,\Delta y,\Delta z)} R_A^t\, \mathbf{R}^t \quad \ldots\ldots (12)$$

where the values of the matrices $\mathbf{R}$ and R_A are easily determined from equations 6 and 2. Although Eq. 12 solves the problem postulated previously, some further clarifications are in order.

In geodesy, due to the peculiar functional relationship between Cartesian and curvilinear geodetic coordinates, it is convenient to define (in general):

$$x_i = \mathcal{F}_i(q_1, q_2, q_3); \quad i = 1,2,3 \qquad (13)$$

where x_i and q_i $(i = 1,2,3)$ represent arbitrary sets of Cartesian and curvilinear coordinates.

Eq. 13 applied to the parameters shown in Fig. 1, takes the form:

$$\begin{Bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{Bmatrix} = \begin{Bmatrix} r \cos \nu \sin \alpha \\ r \cos \nu \cos \alpha \\ r \sin \nu \end{Bmatrix} = \mathbf{R}^t \begin{Bmatrix} 0 \\ 0 \\ r \end{Bmatrix} \qquad (14)$$

were the correspondence $q_1 \equiv \nu$, $q_2 \equiv \alpha$ and $q_3 \equiv r$ is implied.

Then, the Jacobian matrix (hereafter referred to as Jacobian, but termed Jacobi-matrix by some authors possibly to avoid any confusion with the Jacobian determinant) of the transformation is written:

$$\mathrm{J} = \frac{\partial(\Delta e, \Delta n, \Delta u)}{\partial(\nu, \alpha, r)} \qquad (15)$$

The elements of the above matrix may be obtained without taking any partial derivatives by using the equations originally presented in Soler (1976). In particular:

$$J = \mathbf{R}^t H \qquad (16)$$

where $\mathbf{R}$ was defined previously by Eq. 6 and $\mathbf{H}$ may be termed the Lamé matrix, which is diagonal when curvilinear orthogonal coordinates are involved. Namely:

$$H = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} = \begin{bmatrix} r & 0 & 0 \\ 0 & r \cos \nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (17)$$

In this special case, the diagonal elements are the three differential parameters h_i $(i = 1,2,3)$ introduced by Lamé (1840). These parameters are still used in textbooks on mathematical physics, including geodetic references (e.g., Molodenskii, et al. 1960; Heiskanen and Moritz 1975).

The values of the diagonal elements of $\mathbf{H}$ in Eq. 17 were derived in accordance with the ordering of axes and variables presented in this paper (i.e., ν, α, r).

However, when the conventional covariance propagation law is used to compute $\Sigma_{(\nu,\alpha,r)}$ in terms of $\Sigma_{(\Delta e,\Delta n,\Delta u)}$, the explicit inverse functional relationships of ν, α, and r in terms of Δe, Δn, and Δu as given by Eq. 7 are required. Thus, the following Jacobian J' (and not J) should be calculated

$$\mathrm{J}' = \frac{\partial(\nu, \alpha, r)}{\partial(\Delta e, \Delta n, \Delta u)} \qquad (18)$$

J′ is the inverse of the Jacobian given in Eq. 15 or 16. Consequently

$$J' = J^{-1} = H^{-1} R \qquad (19)$$

Therefore, the required Jacobian J′ is nothing more than the rotation matrix **R**, scaled by the inverse of the Lamé matrix $\mathbf{H}^{-1}$.

The above results illustrate the basic difference between the use of Jacobians in lieu of rotation matrices when the covariance law is applied. Rotation matrices are unitless: they conserve the element's units of the original covariance matrix. In this particular example, when Eq. 12 is used, the variance and covariances of the parameters v, α, and r in linear units squared (e.g., cm^2, mm^2) results directly. The corresponding rms errors (i.e., $\sigma_v, \sigma_a, \sigma_d$) are the uncertainties of the position of point B along the v, a, and d axes. The points on the three orthogonal axes (v, a, d) with values σ_v, σ_a, and σ_d belong to the pedal surface of point B, and are not points on the mean error ellipsoid (see Appendix I).

However, when using the standard Jacobian approach, the elements of the resulting covariance matrix will have mixed dimensional units. These will be angular units (radians squared) for the variances σ_v^2 and σ_α^2; the covariance $\sigma_{v\alpha}$, linear units squared for σ_r^2; and mixed units (radians × length) for the covariances σ_{vr} and $\sigma_{\alpha r}$. Thus, in any differential manifold around a point, the transformations between angular and linear units are established along the tangents to the coordinate curves through the Lamé matrix **H**. This matrix, in essence, is playing the role of a "stretching" operator.

From the previous discussion, it follows that dimensional changes between covariance matrices with angular and linear units is achieved through Lamé's matrices. For example, using $\Sigma_{(v,\alpha,r)}$ to denote the covariance matrix in mixed (angular and linear) dimensions and $\Sigma_{(v,a,d)}$ the covariance matrix in linear units of the respectively vertical angle, azimuth, and distance, then:

$$\sum_{(v,a,d)} = \mathbf{H} \sum_{(v,\alpha,r)} \mathbf{H} \qquad (20)$$

and

$$\sum_{(v,\alpha,r)} = \mathbf{H}^{-1} \sum_{(v,a,d)} \mathbf{H}^{-1} \qquad (21)$$

Incidentally, readers familiar with tensor calculus would know that although all Σ matrices in Eqs. 20 and 21 are second rank tensors. Only the matrix $\Sigma_{(v,a,d)}$ contains the so-called tensor physical components. These are the components along the local orthogonal Cartesian frame with axes tangent to the three curvilinear coordinate curves at the point (McConnell 1931). More general tensorial definitions of physical components may be consulted in Altman and de Oliveira (1977). The matrix **H** should be used to transform vector components from angular units (radians) to linear units, for example:

$$\begin{Bmatrix} \sigma_v \\ \sigma_a \\ \sigma_d \end{Bmatrix} = \mathbf{H} \begin{Bmatrix} \sigma_v \\ \sigma_\alpha \\ \sigma_r \end{Bmatrix} \qquad (22)$$

From Eqs. 20, 21 or 22, $\sigma_d = \sigma_r$.

Substituting Eq. 12 into Eq. 21 it follows:

$$\sum_{(\nu,\alpha,r)} = H^{-1}\,\mathbf{R}\,\mathrm{R}_A \sum_{(\Delta x,\Delta y,\Delta z)} \mathrm{R}_A^t\,\mathbf{R}^t\,\mathbf{H}^{-1} \quad (23)$$

Thus, by analogy with the covariance law, it is possible to write:

$$J' = \frac{\partial(\nu,\alpha,r)}{\partial(\Delta x,\Delta y,\Delta z)} = \mathbf{H}^{-1}\,\mathbf{R}\,\mathrm{R}_A \quad (24)$$

$$\text{and} \quad J = (J')^{-1} = \frac{\partial(\Delta x,\Delta y,\Delta z)}{\partial(\nu,\alpha,r)} = \mathrm{R}_A^t\,\mathbf{R}^t\,\mathbf{H} \quad (25)$$

The currently used methodology to determine $\Sigma_{(\nu,\alpha,r)}$ is based on the explicit computation of the partial derivatives of the Jacobian J in Eq. 25 from the functional relationship:

$$\begin{Bmatrix}\Delta x\\ \Delta y\\ \Delta z\end{Bmatrix} = \mathrm{R}_A^t \begin{Bmatrix}\Delta e\\ \Delta n\\ \Delta u\end{Bmatrix} = \mathrm{R}_A^t \begin{Bmatrix} r\cos\nu\sin\alpha\\ r\cos\nu\cos\alpha\\ r\sin\nu\end{Bmatrix} = \mathrm{R}_A^t\,\mathbf{R}^t \begin{Bmatrix}0\\ 0\\ r\end{Bmatrix} \quad (26)$$

The inverse relationship of Eq. 26, that is, formulas expressing ν, α, and r as a function of $(\Delta x, \Delta y, \Delta z)$ necessary to compute J' are presented in the previously cited literature. The required Jacobian follows immediately after the simple matrix multiplication from Eq. 24 without calculating individual Jacobian elements or even the knowledge of the explicit functional relationship (Eq. 26) between parameters. Thus, in practical applications when the covariance matrices are coded in computer programs the explicit analytical form of the Jacobian matrix is not needed. Only the rotational matrices R_A and $\mathbf{R}$ (probably already available, and used previously in the program) are required. The value of the diagonal Lamé matrix $\mathbf{H}$ is straightforward and simple to produce for any set of orthogonal curvilinear coordinates.

In summary, Eqs. 12 and 23 are the two basic formulas for transforming $\Sigma_{(\Delta x,\Delta y,\Delta z)}$ as initially obtained from GPS observations to the final variance-covariance matrix of vertical angle, azimuth, and distance in linear or mixed (angular and linear) units respectively.

Conclusions

In contrast to usual practice, the methodology introduced here provides an alternative approach to determining altazimuthal covariance matrices without computing the Jacobian of the transformations between curvilinear and Cartesian coordinates. The advantage of the method is obvious: the required rotation matrices are easy to compute (products of elementary rotations around the three axes) and generally available by previous transformations. Consequently, no partial derivatives of complex nonlinear functions are involved. In addition, some geometric considerations are clarified when this approach is implemented. For example, the distinction between mean error ellipsoid and pedal surface is uniquely established. The standard deviations of the vertical angle, azimuth and distances (when expressed in linear units) correspond to the lengths along the Cartesian axes (v, a, d) from the forepoint to the pedal surface. Lastly, the problem of transforming angular and linear units between the elements of variance-covariance matrices is simplified by the introduction of the Lamé matrix.

Appendix I.—Pedal Surface Equations

Assume that the symmetric variance-covariance matrix with respect to the (x, y, z) coordinate system is given explicitly by:

$$\sum_{(x,y,z)} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ & \sigma_y^2 & \sigma_{yz} \\ \text{sym} & & \sigma_z^2 \end{bmatrix} \quad (27)$$

Then, the value of the variance (denoted σ_γ^2) along an arbitrary direction of polar angles χ and τ, $(0 \leq \chi \leq 2\pi$, and $-1/2\pi \leq \tau\ 1/2\pi)$, and direction cosines:

$$\{\gamma\} = \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} \cos\tau\cos\chi \\ \cos\tau\sin\chi \\ \sin\tau \end{Bmatrix} \quad (28)$$

may be written: $\sigma_\gamma^2 = \{\gamma\}^t \sum_{(x,y,z)} \{\gamma\}$ .... (29)

For simplicity, assume that $\Sigma_{(x,y,z)}$ is diagonalized and its eigenvalues computed. The resulting diagonal matrix:

$$\sum_{(x_p,y_p,z_p)}^{d} = \begin{bmatrix} \sigma_{x_p}^2 & 0 & 0 \\ & \sigma_{y_p}^2 & 0 \\ \text{sym} & & \sigma_{z_p}^2 \end{bmatrix} \quad (30)$$

can be called the principal variance-covariance matrix. The three eigenvectors associated with Eq. 30 will define the directions of the three principal axes x_p, y_p, z_p with respect to the original (x, y, z) coordinate system.

The values of the principal or uncorrelated standard deviations σ_{x_p}, σ_{y_p} and σ_{z_p} are the magnitudes of the three semi-axes of the principal or mean error ellipsoid.

By definition the "pedal surface" related to a particular error ellipsoid defined by the matrix $\Sigma^d_{(x_p,y_p,z_p)}$ is the locus of points with values σ_{γ_p}, where now the direction cosines are given as in Eq. 28 but with the subscript p (i.e., they refer to the principal axes).

Consequently, using Eq. 29 we may write explicitly:

$$\sigma_{\gamma_p}^2 = s^2 = \{\cos\tau_p\cos\chi_p\ \cos\tau_p\sin\chi_p\ \sin\tau_p\} \sum_{(x_p,y_p,z_p)}^{d} \begin{Bmatrix} \cos\tau_p\cos\chi_p \\ \cos\tau_p\sin\chi_p \\ \sin\tau_p \end{Bmatrix} \quad (31)$$

Thus, in order to express the equation of the pedal surface in Cartesian coordinates, the following substitutes in Eq. 31 must be first implemented:

$$\cos \chi_p = \frac{x_p}{(x_p^2 + y_p^2)^{1/2}}$$

$$\sin \chi_p = \frac{y_p}{(x_p^2 + y_p^2)^{1/2}}$$

$$\cos \tau_p = \frac{(x_p^2 + y_p^2)^{1/2}}{s}$$

$$\sin \tau_p = \frac{z_p}{s}$$

Then, after simplification it is possible to write Eq. 31 as:

$$x_p^2 + y_p^2 + z_p^2 = \left[\left(\frac{x_p}{s}\right) \left(\frac{y_p}{s}\right) \left(\frac{z_p}{s}\right) \right] \sum_{(x_p, y_p, z_p)}^{d} \begin{Bmatrix} \dfrac{x_p}{s} \\ \dfrac{y_p}{s} \\ \dfrac{z_p}{s} \end{Bmatrix} \quad \text{(32)}$$

Finally, the equation of the pedal surface with respect to the principal axes (x_p, y_p, z_p) may be written:

$$(x_p^2 + y_p^2 + z_p^2)^2 - (x_p^2 \sigma_{y_p}^2 + y_p^2 \sigma_{y_p}^2 + z_p^2 \sigma_{z_p}^2) = 0 \quad \text{(33)}$$

Using similar reasoning, Eq. 33 can be generalized for any arbitrary coordinate system (x, y, z) as follows:

$$(\{x\}^t \{x\})^2 = \{x\}^t \sum_{(x,y,z)} \{x\} \quad \text{(34)}$$

In particular, the equation of the pedal surface with respect to the (v, a, d) coordinate system may be obtained after considering that a mapping such as

$$R: (x_p, y_p, z_p) \rightarrow (v, a, d) \quad \text{(35)}$$

always can be established.

Therefore

$$\sum_{(v,a,d)} = \underline{R} \sum_{(x_p, y_p, z_p)}^{d} \underline{R}^t \quad \text{(36)}$$

and defining the coordinates of a point P in the (v, a, d) and (x_p, y_p, z_p) systems respectively by:

$$\{\Delta v\} = \begin{Bmatrix} \Delta v \\ \Delta a \\ \Delta d \end{Bmatrix} \quad \text{(37)}$$

and: $$\{x_p\} = \begin{Bmatrix} x_p \\ y_p \\ z_p \end{Bmatrix} \quad \text{(38)}$$

Finally: $(\{\Delta v\}^t\{\Delta v\})^2 = (\{x_p\}^t\{x_p\})^2 = \{\Delta v\}^t \sum_{(v,a,d)} \{\Delta v\}$ .... (39)

It is easy to demonstrate that the points $(\sigma_v, 0, 0)$, $(0, \sigma_a, 0)$ and $(0, 0, \sigma_d)$ [i.e., the variances along the (v, a, d) frame], belong to the pedal surface because they satisfy Eq. 39.

For example, if:

$$\{\Delta v\} = \begin{Bmatrix} \sigma_v \\ 0 \\ 0 \end{Bmatrix} \qquad (40)$$

then: $$\sigma_v^4 = \{\sigma_v\ 0\ 0\} \begin{bmatrix} \sigma_v^2 & \sigma_{va} & \sigma_{vd} \\ & \sigma_a^2 & \sigma_{ad} \\ \text{sym} & & \sigma_d^2 \end{bmatrix} \begin{Bmatrix} \sigma_v \\ 0 \\ 0 \end{Bmatrix} = \sigma_v^4 \qquad (41)$$

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Appendix III.—Notation

The following symbols are used in this paper:

(e, n, u) = local (right-handed) geodetic coordinate system at any point (λ, ϕ, h). The e axis points to (geodetic) east, n to (geodetic) north and u to the (geodetic) zenith;
H = Lamé matrix;
J = Jacobian matrix;
R = rotation matrix to transform form WGS72 to local geodetic system (e, n, u);
$\mathbf{\bar{R}}$ = rotation matrix to transform from system (e, n, u) to (v, a, d);
r = distance between two points A (standpoint) and B (forepoint);
(v, a, d) = local (right-handed) Cartesian system at any forepoint B; the v axis points to positive ν, a to positive α and d to positive r;

(x, y, z)	=	WGS72 coordinate system as defined by GPS satellite observations;
α	=	geodetic azimuth, $0 \leq \alpha \leq 2\pi$;
λ	=	geodetic longitude, $0 \leq \lambda \leq 2\pi$;
ν	=	vertical angle, $-1/2\pi \leq \nu \leq 1/2\pi$;
ϕ	=	geodetic latitude, $-1/2\pi \leq \phi \leq 1/2\pi$; and
$\Sigma(X, Y, Z)$	=	general variance-covariance matrix of stochastic variables X, Y, Z.