Densifying 3D GPS Networks by Accurate Transformation of Vector Components

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A primary output of Global Positioning System (GPS) post-processed data is a set of non-trivial (independent) vector components and their full covariance information referred to a specific local Cartesian terrestrial frame (e.g., ITRF, WGS84) and epoch. It is important to recognize that when GPS-determined vector components are simultaneously combined into 3D geodetic network adjustments, they should always refer to a common coordinate frame and epoch. This paper uses geometric concepts to formulate rigorous matrix transformations to correct vector components for changes in coordinate systems, secular displacements due to plate rotations, and antenna centering and/or height measuring errors. Finally, the associated variance–covariance matrix of the transformed vector components is derived. *© 2001 John Wiley & Sons, Inc.**

INTRODUCTION

GPS-determined vector components obtained using relative static observations between so-called base (reference) and remote stations are generally grouped by common observing periods termed *sessions.* Attached to each particular session are the date when the observations were taken and the starting and ending GPS time of the observing span. All components of these 3D spatial vectors are referred to a local (topocentric) frame which is parallel to the terrestrial (Earth-fixed) geocentric reference frame defined by the precise ephemeris (e.g., ITRF97, WGS84) selected by the processor at the reduction stage. The epoch of this frame, and actually of the GPS vector components themselves, is the mean epoch of the session observation span, which is always designated by a year and its fraction (e.g., $11^h\ 42^m$ UTC, March 20, 2000 = 2000.2172).

In order to rigorously combine GPS vectors observed at different epochs into a simultaneous least-squares adjustment at epoch t, all vectors belonging to the 3D network must first be transformed to a common epoch and coordinate frame. Furthermore, due to the rotation of the plate on which the GPS stations are located, the geocentric positions of the two points defining each vector have changed since epoch t_0 when the original GPS observations were taken, to the epoch t selected for the 3D adjustment. The influence of plate rotations on GPS vectors processed at different epochs cannot be ignored if accurate GPS results are expected. The relative motions between points along subduction zones could be as large as 24 cm/yr (Bevis et al., 1995), the fastest crustal motions yet observed, which are easily detectable using modern GPS techniques. However, none of the transformations detailed here are addressed in standard GPS text books (e.g., Leick, 1995; Hofmann-Wellenhof et al., 1997). Soler (1998) initially presented equations to transform vector components between two arbitrary reference frames from epoch t_0 to epoch t by taking into account their differences in orientation and scale, as well as the motion of the plates where the points are located. Since the vector components are always given with respect to local terrestrial frames with origin at the base station, the possible shifts (T_x, T_y, T_z) between the origins of the various conventional geocentric frames do not even enter into the formulation. Nevertheless, previously undetected small displacements δx, δy, and δz, along the three Cartesian components, caused by possible antenna centering and/or height

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measuring errors, could have been discovered when the station was reoccupied at epoch t. This makes the position of the antenna at time t not exactly the same as that at time t_0. An extreme case of this problem could be the misidentification of a site mark involved in two surveys carried out at epochs t_0 and t.

This situation is more common than it appears at first glance. Until recently there was not total agreement among GPS users on which point should be selected as the primary reference point on an antenna when processing carrier-phase observables. It was general practice not long ago to assume that the L1 phase center was the best point to tie the observations to the site, usually defined by a brass disk or reference mark. This thinking has changed in recent years and currently the "antenna reference point" (ARP), which is physically located at the geometric center of the bottom surface of the antenna, is considered the logical reference point of the antenna. The main argument favoring this preference alludes to the fact that the spatial position of the L1 phase center is not a well-defined electronic point, since it changes position as a function of the incoming GPS signal and the electrical characteristics of the antenna. This was empirically corroborated as a result of several investigations that modeled antenna-phase-center patterns (e.g., Mader & MacKay, 1996; Meertens et al., 1996; Rothacher & Schär, 1996). Failure to account for antenna-phase-center variation can lead to errors of up to 10 cm in height when processing GPS data for a baseline involving two different antenna types.

Another term frequently quoted in GPS literature is that of "antenna parameters." This applies to the various constants peculiar to each individual antenna, establishing the relationship between the fundamental hardware elements, e.g., nominal phase centers L1 and L2, ARP, ground plane, etc. These quantities are provided by the antenna manufacturer or otherwise should be precisely calibrated by the user. The National Geodetic Survey (NGS) has calibrated most GPS geodetic antennas using a methodology described by Mader (1999). Diagrams of GPS antennas and their calibrated parameters can be accessed at the following Web address: http://www.grdl.noaa.gov/ GRDL/GPS/ Projects/ ANTCAL/.

One clarification is in order, the ARP is not necessarily the "station reference point (SRP)." The SRP is more often than not the center of the physical disk attached to a steel pipe buried in concrete in the ground and used to permanently mark the location of the station. In classical geodesy and/or surveying practice, the SRP is traditionally the so-called "monument." This is a logical choice considering that it is the only remaining permanent marker once the observations are completed and the antenna replaced or removed from the site. Thus, when a permanent mark is available, all GPS observations should be reduced to this mark. However, many GPS "fiducial" stations do not have ground marks per se and, consequently, the ARP is assumed to be coincident with the SRP. If that is the case, the antenna height, i.e., the distance between ARP and SRP, is zero. The term "fiducial" is loosely applied to name continuously operating GPS sites whose RINEX2 data are made available electronically to the GPS community. Examples include the NGS Continuously Operating Reference Stations (National CORS) network (Snay & Weston, 1999) or the International GPS Service (IGS) global network of permanent GPS trackers [http://igscb.jpl.nasa.gov]. All fiducial stations provide access to station logs where information about its occupation history is available. This includes the different type of antennas used during the years, the adopted antenna constants, ARP height over the mark if any, ties to nearby points at the site, etc.

THEORY

Soler (1998) followed an algebraic reasoning to obtain rigorous transformation of GPS-determined vector components between coordinate frames and epochs. However, the influence of antenna centering and/or height setup errors were not discussed. This was expanded in Soler et al. (1999), although no equations to determine the variance–covariance matrices of the transformed vector components were presented. In contrast, the present paper introduces a novel rigorous matrix solution strictly based on geometric concepts, followed by general equations that incorporate the variance–covariance matrix of transformed vector components.

Figure 1 shows the basic notation convention adopted through this work to identify vector components. Assume a base station A and a remote station B defining a GPS vector between points A and B observed at time t_0. The components of vector $\overrightarrow{AB}$ will be denoted in matrix form by the column vector $\{\Delta x(t_0)\}$ or explicitly:

$$\begin{Bmatrix} \Delta x(t_0) \\ \Delta y(t_0) \\ \Delta z(t_0) \end{Bmatrix} = \begin{Bmatrix} x_B(t_0) - x_A(t_0) \\ y_B(t_0) - y_A(t_0) \\ z_B(t_0) - z_A(t_0) \end{Bmatrix} \tag{1}$$

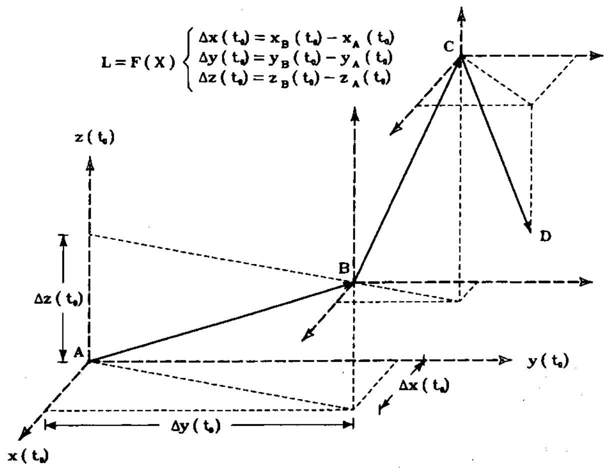

FIGURE 1. Vector components and time t_0 and their explicit mathematical model.

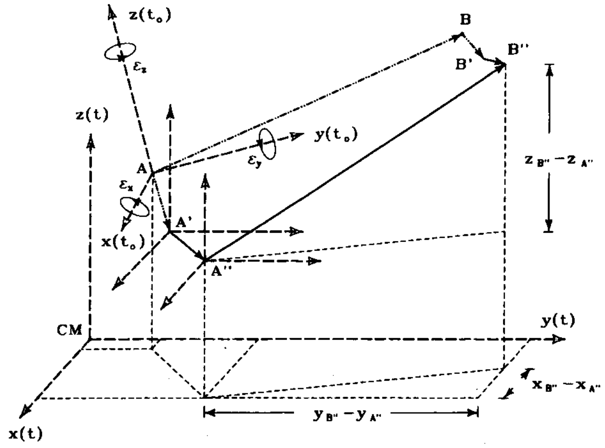

FIGURE 2. Transformation of vector components between two frames at time t_0 and t.

The same logic will apply to any another vector, e.g., between B and C except that, as Figure 1 shows, the local terrestrial frame at B is assumed parallel to the local frame at A. Rigorously speaking this is only true if the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ were observed during the same session. If, for example, vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ were measured on different GPS campaigns, their components do not refer to the same epoch; furthermore, they may not even refer to the same coordinate frame. Figure 1 also shows the well-known mathematical model typically used on GPS networks least-squares adjustments. The observables are the vector components at some specific epoch, say t_0, and the parameters (unknowns) are the coordinates of the points involved at the same epoch. Recall that because some vectors are reobserved, or because there are many vectors starting or ending at the same station, the redundancy of the adjustment is assured.

Assume now that one wants to transform vector components from frame ITRFyy at epoch t_0 to frame ITRFzz at epoch t. This transformation could be designated symbolically by the mapping ITRFyy(t_0) $\rightarrow$ ITRFzz(t). Assume further that one wants to correct for antenna errors (decentering and/or height) detected at epoch t_0 long after the original GPS vector components were processed and archived. According to Figure 2 the assumed antenna displacement errors are represented by vectors $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ at points A and B respectively. Notice that in the figure, all vectors referred to epoch t_0 are drawn using the same line style (broken line separated by two points). Similarly, one also may assume antenna collimation and height errors at time t. They are represented in the figure as vectors $\overrightarrow{A'A''}$ and $\overrightarrow{B'B''}$

Consequently, the original components of vector $\overrightarrow{AB}$ processed and archived at epoch t_0 should be replaced by vector $\overrightarrow{A''B''}$ at epoch t (see Figure 2) before it is combined in a network adjustment with other new GPS vectors measured at t. As Figure 2 depicts, the frames at t_0 and t are assumed not parallel. Moreover, they may even have different scales. From the figure and by simple geometric and vector considerations, one can arrive at the following equality:

$$\overrightarrow{A''B''}(t) = -\overrightarrow{A'A''}(t) - \overrightarrow{AA'}(t_0) + \overrightarrow{AB}(t_0) + \overrightarrow{BB'}(t_0) + \overrightarrow{B'B''}(t) \tag{2}$$

Grouping terms by epochs,

$$\overrightarrow{A''B''}(t) = \overrightarrow{AB}(t_0) + \overrightarrow{BB'}(t_0) - \overrightarrow{AA'}(t_0) + \overrightarrow{B'B''}(t) - \overrightarrow{A'A''}(t) \tag{3}$$

To distinguish between the vector components, previously denoted with the symbol Δ, and the antenna displacements—i.e., misalignments and/or height errors—the latter will be identified, as mentioned above, with the symbol δ. Figure 2 also shows that, in the most general case, for each vector $\overrightarrow{AB}$ there are four possible errors associated with antenna displacements. These may be located at the two points A and B, each of them spanning two possible epochs t_0 and t. Thus, in addition to the epoch, it is important to identify the point in question, e.g., for the displacement AA′ at epoch t_0 the compact vector matrix notation $\{\delta x_A(t_0)\}$ will be used. This can be written explicitly:

$$\begin{Bmatrix} \delta x_A(t_0) \\ \delta y_A(t_0) \\ \delta z_A(t_0) \end{Bmatrix} = \begin{Bmatrix} x_{A'}(t_0) - x_A(t_0) \\ y_{A'}(t_0) - y_A(t_0) \\ z_{A'}(t_0) - z_A(t_0) \end{Bmatrix} \tag{4}$$

Similar equations apply to other possible antenna displacements at points B, A′, and B′, denoted respectively by $\{\delta x_B(t_0)\}$, $\{\delta x_{A'}(t)\}$, and $\{\delta x_{B'}(t)\}$. However, in practice the antenna displacements at points A, B, A′, and B′ are known along local geodetic horizon frames (east≡e, north≡n, up≡u). For example, assume that the antenna at station A at epoch t_0 was perfectly centered, but when the vectors were processed—say, more than 5 years ago—the L1 phase center and not the ARP was used as SRP. Because the new vector at epoch t refer to the ARP, a correction (from L1 to ARP) must be applied to the old vector components. Furthermore, assume for the sake of discussion that the model of antenna used was a Dorne Margolin T. It is known that for this particular type of antenna the ARP is located 11 cm below the L1 phase center. Because the antenna was leveled at the time the observations were originally collected, this distance is aligned with the direction of the plumb line which, neglecting vertical deflections, coincides with the normal to the adopted reference ellipsoid. Thus, the 11 cm are actually counted along the ellipsoid normal. Using a notation similar to Eq. (4) but referred to a local geodetic horizon frame (e, n, u), one can write the vector components along this frame as $\{\delta e_A(t_0)\}$, or explicitly for this specific example:

$$\begin{Bmatrix} \delta e_A(t_0) \\ \delta n_A(t_0) \\ \delta u_A(t_0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -11\text{cm} \end{Bmatrix} \tag{5}$$

The negative sign of the height correction is explained by the fact that the location of the ARP in Dorne Margolin T antennas is located below its L1 phase center. Nevertheless, corrections to GPS vector components should be expressed on a local frame which is not local geodetic, as Eq. (5) implies, but a local frame parallel to the geocentric terrestrial frame. The transformation from local horizon to local terrestrial frames is achieved through the rotation matrix:

$$\begin{aligned}[R] &= R_3(-\lambda - \tfrac{1}{2}\pi)R_1(\varphi - \tfrac{1}{2}\pi) \\ &= R_3(-\lambda)R_2(\varphi - \tfrac{1}{2}\pi)R_3(-\tfrac{1}{2}\pi)\end{aligned} \tag{6}$$

or explicitly

$$[R] = \begin{bmatrix} -\sin\lambda & -\cos\lambda\sin\varphi & \cos\lambda\cos\varphi \\ \cos\lambda & -\sin\lambda\sin\varphi & \sin\lambda\cos\varphi \\ 0 & \cos\varphi & \sin\varphi \end{bmatrix} \tag{7}$$

The above rotation matrix should always be computed at some specific point and epoch. Thus, the antenna displacements at point A, epoch t_0, along the local terrestrial frame is determined from (5) according to the transformation:

$$\begin{Bmatrix} \delta x_A(t_0) \\ \delta y_A(t_0) \\ \delta z_A(t_0) \end{Bmatrix} = [R]_{A_{t_0}} \begin{Bmatrix} \delta e_A(t_0) \\ \delta n_A(t_0) \\ \delta u_A(t_0) \end{Bmatrix} \tag{8}$$

Thus, replacing the vectors in Eq. (3) by their equivalent matrix columns after introducing the required scale factor and the rotation matrix to make the geocentric frame at t_0 parallel to the geocentric frame at t, one arrives at the following matrix equation:

$$\begin{aligned}\begin{Bmatrix} \Delta x(t) \\ \Delta y(t) \\ \Delta z(t) \end{Bmatrix} &= (1+s)\begin{bmatrix} 1 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{bmatrix} \times \left\{ \begin{Bmatrix} \Delta x(t_0) \\ \Delta y(t_0) \\ \Delta z(t_0) \end{Bmatrix} \right. \\ &\quad \left. + \begin{Bmatrix} \delta x(t_0) \\ \delta y(t_0) \\ \delta z(t_0) \end{Bmatrix} \right\} + \begin{Bmatrix} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{Bmatrix}\end{aligned} \tag{9}$$

where,

$$\begin{aligned}\begin{Bmatrix} \delta x(t_0) \\ \delta y(t_0) \\ \delta z(t_0) \end{Bmatrix} &= \begin{Bmatrix} \delta x_B(t_0) - \delta x_A(t_0) \\ \delta y_B(t_0) - \delta y_A(t_0) \\ \delta z_B(t_0) - \delta z_A(t_0) \end{Bmatrix} \\ &= [R]_{B_{t_0}}\{\delta e_B(t_0)\} - [R]_{A_{t_0}}\{\delta e_A(t_0)\}\end{aligned} \tag{10}$$

and similarly,

$$\begin{aligned}\begin{Bmatrix} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{Bmatrix} &= \begin{Bmatrix} \delta x_{B'}(t) - \delta x_{A'}(t) \\ \delta y_{B'}(t) - \delta y_{A'}(t) \\ \delta z_{B'}(t) - \delta z_{A'}(t) \end{Bmatrix} \\ &= [R]_{B'_t}\{\delta e_{B'}(t)\} - [R]_{A'_t}\{\delta e_{A'}(t)\}\end{aligned} \tag{11}$$

In Eq. (9), the variables ε_x, ε_y, and ε_z (expressed in radians) are the differential rotations about the axes of the ITRFyy frame required to make it parallel to ITRFzz. Counterclockwise (anticlockwise) rotations are assumed positive. The parameter s (unitless in ppm $\times 10^{-6}$) is the differential scale factor required to change the unit of scale of the ITRFyy frame to make it consistent with the ITRFzz frame. Table 1 shows the latest transformation parameters and their standard deviations between geocentric frames, which realizations were attained using the ephemerides of the GPS (ITRF and WGS84) and GLONASS (PZ-90) satellite constella-

TABLE 1

Transformation parameters and their standard deviations between modern geocentric frames

	T_x cm	T_y cm	T_z cm	ϵ_x mas	ϵ_y mas	ϵ_z mas	s ppb
PZ-90 → WGS84 (G873)[1]	−108.0 ± 21.0	−27.0 ± 21.0	−90.0 ± 33.0	0.0	0.0	−160.0 ± 10.0	−120.0 ± 60.0
WGS84 (G873) → ITRF94[2]	9.6 ± 5.5	6.0 ± 5.5	4.4 ± 5.4	−2.2 ± 2.1	−0.1 ± 2.1	1.1 ± 2.2	−14.3 ± 8.4
ITRF94 ≡ ITRF96 → ITRF97[3]	0.03 ± 0.21	0.05 ± 0.21	−1.47 ± 0.21	−0.159 ± 0.900	0.263 ± 0.098	0.060 ± 0.088	1.43 ± 0.31

NOTES. mas ≡ milliarc second; ppb ≡ parts per billion = 10^{-3} ppm. [1]epoch ~1997.0 (Bazlov et al., 1999a, 1999b); [2]epoch 1997.0 (Malys et al., 1997); [3]epoch 01-AUG-1999 (IGS e-mail #2432). The equivalence between ITRF94 and ITRF96 is mentioned in (Sillard et al., 1998).

tions. The tabulated values clearly show that the geocentricity of the GLONASS system is not yet accurately determined.

A generalization of Eq. (9) can be written introducing the velocities of points A and B at time t_0, denoted by v_A and v_B (generally given in meters per year), caused by plate motion or any other known secular tectonic displacement which was previously neglected. Thus, finally:

$$\begin{Bmatrix} \Delta x(t) \\ \Delta y(t) \\ \Delta z(t) \end{Bmatrix} = (1+s) \begin{bmatrix} 1 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{bmatrix} \times \left\{ \begin{Bmatrix} \Delta x(t_0) \\ \Delta y(t_0) \\ \Delta z(t_0) \end{Bmatrix} + \begin{Bmatrix} \delta x(t_0) \\ \delta y(t_0) \\ \delta z(t_0) \end{Bmatrix} + (t-t_0) \begin{Bmatrix} v_{B_x} - v_{A_x} \\ v_{B_y} - v_{A_y} \\ v_{B_z} - v_{A_z} \end{Bmatrix} \right\} + \begin{Bmatrix} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{Bmatrix} \tag{12}$$

Predicted velocities at any location in the United States can be obtained interactively from the NGS web site [http:// www.ngs.noaa.gov] by clicking on "Products and Services" and then "HTDP—Horizontal Time-Dependent Position." To learn more about HTDP, the reader can consult Snay (1999).

VARIANCE–COVARIANCE MATRIX OF THE TRANSFORMED COMPONENTS

Usually, the final components of each vector in a particular GPS session are accompanied by their full variance–covariance matrix or, equivalently, by their standard errors and correlation matrix. This is the adopted NGS format, the so-called GFILE, used from the early 1980s to archive processed vector components for every GPS project. The GFILE of a typical GPS project contains many sessions. Thus, to transform GFILEs between frames and epochs, not only the vector components, but also their corresponding full variance–covariance matrix could be transformed. The required input values are the original variance–covariance matrix of the vector components at t_0, the variance–covariance matrix of the rotations and scale that makes the frame at t_0 parallel to the frame at t and, finally, the variance–covariance matrix of the velocities at t_0. In the discussion that follows it will be assumed that the antenna displacements, if any, are errorless with no standard errors attached to them. This is the common situation when antenna height offset errors implicit in old field procedures are subsequently detected or when a change of antenna height parameters is implemented.

Because in a particular GPS session one may have more than a single vector, the notation of Eq. (12) will be abbreviated and extended to identify at least two such vectors (six components) denoted with the sub-indices i, j. Thus,

$$\{\Delta\}_i = \{\Delta x(t_0)\}_i;\ \{\Delta\}_j = \{\Delta x(t_0)\}_j;\ \text{etc.}$$
$$\{v\}_i = \{v_B - v_A\}_i;\ \{v\}_j = \{v_D - v_C\}_j;\ \text{etc.}$$

The following compact notation is introduced to simplify the formulation as much as possible. $[\delta\Re]$ is the differential rotation matrix of the transformation between the frames ITRFyy(t_0) and ITRFzz(t), namely,

$$[\delta\Re] = \begin{bmatrix} 1 & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{bmatrix} \tag{13}$$

In what follows $[\underline{\Delta}]_i$ and $[\underline{\nu}]_i$ indicate skew-symmetric (antisymmetric) matrices, respectively, of the components of vector i and the differences between the velocities of points B and A:

$$[\underline{\Delta}]_i = \begin{bmatrix} 0 & -\Delta z(t_0) & \Delta y(t_0) \\ \Delta z(t_0) & 0 & -\Delta x(t_0) \\ -\Delta y(t_0) & \Delta x(t_0) & 0 \end{bmatrix}_i \tag{14}$$

$$[\underline{\nu}]_i = \begin{bmatrix} 0 & -(\nu_{B_z} - \nu_{A_z}) & \nu_{B_y} - \nu_{A_y} \\ \nu_{B_z} - \nu_{A_z} & 0 & -(\nu_{B_x} - \nu_{A_x}) \\ -(\nu_{B_y} - \nu_{A_y}) & \nu_{B_x} - \nu_{A_x} & 0 \end{bmatrix}_i \tag{15}$$

With the above notation, the Jacobian contribution of the functional relationship (12), due only to effects of rotations and scale, can be reduced to the following matrix expression:

$$\underset{(3\times 4)}{[J_{\varepsilon s}]_i} = [(1+s)[[\underline{\Delta}]_i + (t-t_0)[\underline{\nu}]_i] \,\vdots\, [\delta\Re]\{\Delta\}_i + (t-t_0)\{\nu\}_i] \tag{16}$$

Using the nomenclature just introduced, the transformed variance–covariance matrix of the components of an arbitrary vector i can be written:

$$\Sigma_{\Delta(t)_i} = (1+s)^2\,[\delta\Re]\,\Sigma_{\Delta(t_0)_i}\,[\delta\Re]^t + [J_{\varepsilon s}]_i\,\Sigma_{\varepsilon s}\,[J_{\varepsilon s}]_i^t + (1+s)^2(t - t_0)^2[\delta\Re][\Sigma_{\nu_A} - \Sigma_{\nu_A\nu_B} + \Sigma_{\nu_B} - \Sigma_{\nu_B\nu_A}][\delta\Re]^t \tag{17}$$

The variance–covariance matrix of the transformed vector components at epoch t of a session with n vectors will have $n(3\times 3)$ diagonal blocks. An arbitrary block i will be computed according to Eq. (17).

The cross-covariance between the components of two arbitrary vectors i, j, respectively, between points A, B, and C, D, can be written as follows:

$$\Sigma_{\Delta(t)_i\Delta(t)_j} = (1+s)^2\,[\delta\Re]\,\Sigma_{\Delta(t_0)_i\Delta(t_0)_j}\,[\delta\Re]^t + [J_{\varepsilon s}]_i\,\Sigma_{\varepsilon s}\,[J_{\varepsilon s}]_j^t + (1+s)^2(t-t_0)^2\,[\delta\Re]\,[\Sigma_{\nu_A\nu_C} - \Sigma_{\nu_A\nu_D} + \Sigma_{\nu_B\nu_D} - \Sigma_{\nu_B\nu_C}][\delta\Re]^t \tag{18}$$

In the above equations the variance–covariance matrix of the rotations and scale is given as usual by:

$$\Sigma_{\varepsilon s} = \begin{bmatrix} \sigma^2_{\varepsilon_x} & \sigma_{\varepsilon_x\varepsilon_y} & \sigma_{\varepsilon_x\varepsilon_z} & \sigma_{\varepsilon_x s} \\ \sigma_{\varepsilon_y\varepsilon_x} & \sigma^2_{\varepsilon_y} & \sigma_{\varepsilon_y\varepsilon_z} & \sigma_{\varepsilon_y s} \\ \sigma_{\varepsilon_z\varepsilon_x} & \sigma_{\varepsilon_z\varepsilon_y} & \sigma^2_{\varepsilon_z} & \sigma_{\varepsilon_z s} \\ \sigma_{s\varepsilon_x} & \sigma_{s\varepsilon_y} & \sigma_{s\varepsilon_z} & \sigma^2_s \end{bmatrix} \tag{19}$$

while the cross-covariance of, e.g., the velocities of points A and C is:

$$\Sigma_{\nu_A\nu_C} = \begin{bmatrix} \sigma_{\nu_{A_x}\nu_{C_x}} & \sigma_{A_x\nu_{C_y}} & \sigma_{\nu_{A_x}\nu_{C_z}} \\ \sigma_{\nu_{A_y}\nu_{C_x}} & \sigma_{\nu_{A_y}\nu_{C_y}} & \sigma_{\nu_{A_y}\nu_{C_z}} \\ \sigma_{\nu_{A_z}\nu_{C_x}} & \sigma_{\nu_{A_z}\nu_{C_y}} & \sigma_{\nu_{A_z}\nu_{C_z}} \end{bmatrix} \tag{20}$$

Similar logic can be applied to the other variance–covariance matrices needed in Eqs. (17) and (18). Notice that while variance–covariance matrices are symmetric, that is not necessarily the case for cross-covariances. However, the following identities are fulfilled:

$$\Sigma_{\Delta(t)_i\Delta(t)_j} = \Sigma^t_{\Delta(t)_j\Delta(t)_i};\ \Sigma_{\varepsilon s\nu_A} = \Sigma^t_{\nu_A\varepsilon s};\ \text{etc.} \tag{21}$$

In practical situations each session cross-covariances between the vector components and velocities, vector components and transformation parameters, and transformation parameters and velocities are not known and were assumed zero above, i.e.,

$$\Sigma_{\Delta(t_0)_i\nu_A} = \Sigma_{\Delta(t_0)_i\varepsilon s} = \Sigma_{\varepsilon s\nu_A} = \text{etc.} = [0] \tag{22}$$

If that is not the case, the following term should be added to Eq. (17):

$$(1+s)[[A] + [A]^t + (t-t_0)[[B] + [B]^t + (1+s)[[\delta\Re] \times [[C] + [C]^t - [D] - [D]^t]][\delta\Re]^t]]] \tag{23}$$

where:

$$[A] = [J_{\varepsilon s}]_i\,\Sigma_{\varepsilon s\Delta(t_0)_i}[\delta\Re]^t \tag{24}$$

$$[B] = [J_{\varepsilon s}]_i\,[\Sigma_{\varepsilon s\nu_B} - \Sigma_{\varepsilon s\nu_A}]\,[\delta\Re]^t \tag{25}$$

$$[C] = \Sigma_{\Delta(t_0)_i\nu_B} \tag{26}$$

and

$$[D] = \Sigma_{\Delta(t_0)_i \nu_A} \tag{27}$$

Similarly, the following expression should be added to Eq. (18):

$$(t - t_0)(1 + s)^2[\delta\Re][\Sigma_{\Delta(t_0)_j \nu_B} - \Sigma_{\Delta(t_0)_j \nu_A} + \Sigma_{\Delta(t_0)_i \nu_D} - \Sigma_{\Delta(t_0)_i \nu_C}]$$
$$[\delta\Re]^t + (1 + s)[J_{\varepsilon s}]_i[\Sigma_{\varepsilon s \Delta(t_0)_j} + (t - t_0)[\Sigma_{\varepsilon s \nu_D} - \Sigma_{\varepsilon s \nu_C}]][\delta\Re]^t$$
$$+ (1 + s)[\delta\Re][\Sigma_{\Delta(t_0)_i \varepsilon s} + (t - t_0)[\Sigma_{\nu_B \varepsilon s} - \Sigma_{\nu_A \varepsilon s}]][J_{\varepsilon s}]_j^t \tag{28}$$

CONCLUSIONS

This article introduces rigorous matrix equations to transform GPS-determined vector components and their variance–covariance matrices from one arbitrary frame at epoch t_0 to another at epoch t. The methodology presented also assumes knowledge of the velocities of the points involved, as well as possible antenna set up errors. Finally, the article introduces novel accurate transformation of vector components variance-covariance matrices without neglecting any of the possible cross-correlations between the different parameters involved. ■

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